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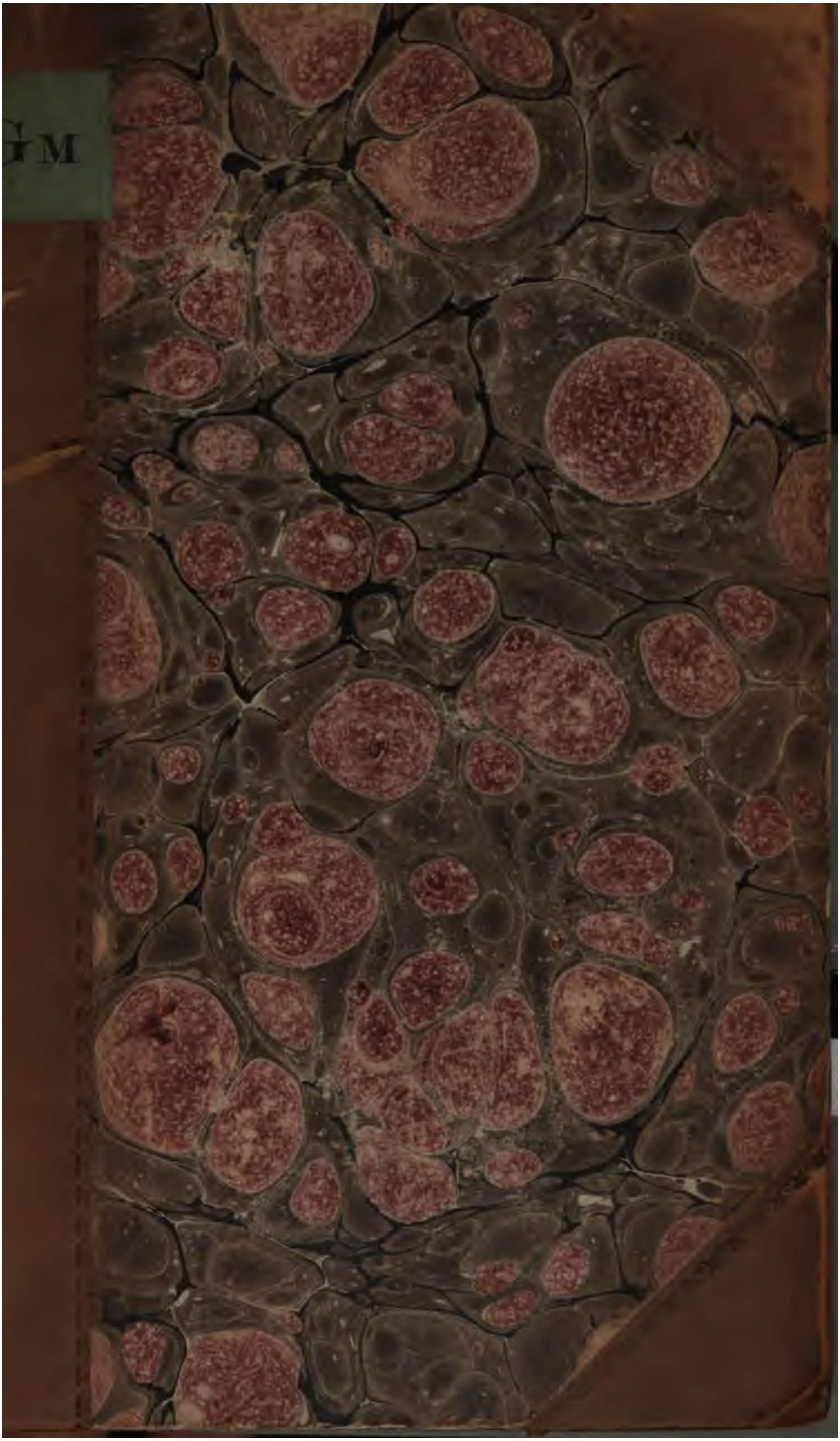

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1826

ELEMENTS

OF

ANALYTIC TRIGONOMETRY,

PLANE AND SPHERICAL.

BY

F. R. HASSLER, F. A. P. S.

New-York:

PUBLISHED BY THE AUTHOR.

James Bloomfield, Printer.

1826

799

Southern District of New-York, ss.

BE IT REMEMBERED, that on the tenth day of June, A. D. 1826, in the fiftieth year of the Independence of the United States of America, F. R. Hassler, of the said District, hath deposited in this office the title of a Book the right whereof he claims as Author, in the words following, to wit:—

(L.S.)

"Elements of Analytic Trigonomety, Plane and Spherical." By F. R. Hassler, F. A. P. S."

In conformity to the Act of Congress of the United States, entitled, "An Act for the encouragement of Learning, by securing the copies of Maps, Charts, and Books, to the authors and proprietors of such copies, during the time therein mentioned." And also to an Act, entitled "An Act, supplementary to an Act, entitled an act for the encouragement of Learning, by securing the copies of Maps, Charts, and Books, to the authors and proprietors of such copies, during the times therein mentioned, and extending the benefits thereof to the arts of designing, engraving, and etching historical nnd other prints."

JAMES DILL,
Clerk of the Southern District of New-York.

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REMARK:—As it is essential, in an elementary book, that the expressions be appropriate to the subject, and even to the local usages of the language of the science, of whose elements it treats, as far as the latter can be admitted without diminishing the precision of the expressions; and as this would require the author to be a native of the country in whose language the treatise is published, which is not my case; my friend Professor Renwick, so advantageously known to the public by his own works, has done me the favour to translate into English the manuscript of this work, which I drew up in French. We considered this as the surest means of obtaining the desired object of bringing this work before the public in a style unembarrassed by other idioms, and whose expressions would be adapted, not only to the language itself, but to established usages of this science.

INTRODUCTION.

MATHEMATICAL science must, from its very nature, have taken its rise in the simple inspection of geometric figures. The abstractions, upon which the calculus is founded, and whose great extension and generalization has produced the analytic method, must have arisen at a later period, as the product of a higher cultivation of the powers of the mind.

During the period that geometry constituted the principal part of mathematical science, trigonometry was necessarily treated of by the synthetic methods applicable to that branch of the science; and the solution of its several problems, attained by mere construction. Calculation was subsequently introduced, when the means were discovered, by which numbers could be applied to express the relations of quantities, which appear so different in their respective natures, as linear dimensions and angles.

Analysis, so bold in its steps and so universal in its methods, which has carried mathematical science to results the most general, and of such extensive and useful consequences, has naturally changed the mode of proceeding in trigonometry, as well as in other departments of mathematics. It is therefore necessary now, in order to study trigonometry in a truly scientific way, to treat of it in the most general manner; and, proceeding from principles the most general, yet at the same time the most simple and elementary, to found upon them a complete system; whose results may be fitted for universal application.

It is not necessary to enter into all the details, that are the necessary consequences of such a system, in endeavouring to attain this object; they will not escape the researches of him, who has made himself master of the system itself.

With such views the present elementary treatise has been drawn up; and it is not necessary to explain the difference, that exists between it and the various other manners in which trigonometry has been treated of.* The principles upon which it is grounded are the following.

As straight lines and angles, or portions of the circumference of a circle, are incommensurable quantities, they cannot be directly compared. But the ratio between two of the sides of a right angled triangle, will determine the magnitude of the acute angles; the third angle being always given, in consequence of the primitive condition of rectangularity in the triangle. This ratio then is the true and only means by which angles may be compared with straight lines.

The names that are given to the several ratios, that exist among the sides of a right angled triangle, taken by pairs, are purely conventional, although the terms have in part been deduced from geometric considerations, having reference to the circle. But it is of the greatest importance carefully to avoid confounding the lines, that correspond to these ratios, or trigonometrical functions, when represented in a circle, with these ratios themselves.†

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* It was the desire of introducing into the course of mathematics at the United States' military academy at West-point, the most useful mode of instruction in this branch, that led me to the preparation of this work, as early as the year 1807.

† The term *sine* owes its origin simply to a contraction in writing *semissis cordæ;* when, in the middle age, instead of the chords of angles, that were formerly employed in calculation, their halves were introduced, writing merely *sm;* and *co-sm* for *complementi semissis cordæ*, the tangent is represented geometrically by the line touching the circle without cutting it, and is the only appropriate denomination taken from the circle. The prolongation of the radius, until it cut the tangent, has been called *secant*, which is a perversion of the name given in geometry to a line that cuts the circle without passing through the centre. The addition, *co*, before each of these names, refers them, as in the case of the sine, to the complementary angle.

Setting out then from the primary definitions of the ratios, that exist between the three lines that form a right angled triangle, combining them, simply, and by their squares, according to the properties of right angled triangles, deduced from the most elementary geometry; (Euclid I. p. 47) we shall obtain, by means of the four fundamental rules of ordinary arithmetic, applied algebraically to these elementary expressions, a series of elementary formulæ.

These formulæ give the solution of every possible case of right lined rectangular trigonometry; and furnish a general table for the reduction of the several trigonometric functions to each other; similar in its nature and application to the common multiplication table. In this way we are furnished with a system of quantities, whose relative relations are determined; the fruitful source of every possible combination.

By the simple consideration of two angles united by juxta or super-position, (a method employed in elementary geometry,) applying the same elementary process, founded upon the principles previously employed, the second step in the system is made. This step furnishes the general principles of the combination of the trigonometric functions of the sum and difference of two or more angles.

The same system of combination used before, applied to this second series of formulæ; with different assumptions in relation to the relative value of the two angles; and also when they are supposed to have a constant determinate value; leads to all the various formulæ that can be desired; which are given in regular tables systematically arranged; and which may be referred to with the greatest readiness.

This mode of proceeding appears to lead to the desired aim with the least labour of intellect, and thus in the most easy way to the final end; which is, to present to the reader a full system of this branch of mathematics, in such a way, as to furnish every necessary element for the solutions of trigonometry, both plane and spherical; and for the use of analysis in general, in its numerous applications to geometry, and to transcendant quantities.

It is with a similar view that the chapter which points out the mode of making use of the trigonometric functions in the integral calculus, and chiefly for the purpose of transforming the formulæ to fit them for integration, has been inserted. Trigonometric differentials are however omitted; they would require the application of the differential calculus, the knowledge of which is not to be presumed in the student of elementary trigonometry. It was thought more expedient to defer this part to a subsequent extension of the course of trigonometry; that should at the same time present its applications, and several other problems; both theoretic and practical, (and which will form the sequel of this elementary treatise, if it be approved by the public.)

It is thought: that the method of applying the trigonometric functions to algebra, by a change of the formulæ, such as to admit the use of logarithms, to change addition or subtraction into multiplication, &c. a method as simple as useful, is sufficiently explained by the use which is made of it in the course of this treatise. For this reason it has not been separately considered, as it might have been, in applying it to the solution of equations of the second and third order, &c. But when the applications, that are actually made of it in this treatise, are well understood, those to other cases will be also intelligible.

Although, for the reasons already stated, the explanation of the ingenious methods, that may be employed in the construction of trigonometric tables, both natural and logarithmic, is not admitted into this plan; it has been thought proper to explain their fundamental principles; in order to complete the system.

The considerations that have reference to the radius of the circle, are not given, except where it becomes necessary to employ them; thus the student does not find himself embarrassed with them in those parts, where they could answer no other purpose, but that of confusing his ideas.

These principles being established, the solution of all the cases of oblique angled plane triangles follows, as their most obvious application; and the use that is made of the forms that are given to the trigonometric functions, in reducing the calculations to logarithms, is a sufficient introduction to this method.

When the analytical method is applied to spherical trigonometry, it is obviously proper, first to expose some of the immediate consequences of the theorems of solid geometry, in their application to the sphere, and then to express them in the form of trigonometric functions. Setting out in this manner immediately from solid geometry, we avoid, as will be seen, all the delay and difficulty, which would attend the introduction of spherics in the abstract.

The combinations of the parts of the right angled triangles, that constitute the elements of a spherical triangle, considered by the method of trigonometric functions, also forms in this part of the work the principle whence all the elementary formulæ of spherical trigonometry are deduced. The combinations of these give all the solutions, that this branch of trigonometry demands.

It has been thought that the continuation of the method previously used, was also in this part of trigonometry preferable to the introduction of another, although equally good in itself; for it is with methods in mathematics, as with style in ordinary writings: that author is most easily understood, who expresses himself in one uniform and fixed manner; while a change in the method of expression naturally introduces uncertainty in the apprehension of the sense of the writer.

For a similar consideration, the means of deduction, or the representation of the different subjects, have not been multiplied: an elementary book need not give all that the author knows on the subject, but only all that is necessary to constitute a complete system.

The mechanical arrangement of a calculation may conduce to its accuracy, and to the ease of revising it, in case of need. It is with this, as with order in all matters of business; it is proper in the beginning to acquire good habits, which practice will render easy. The numerous and frequently complicated operations of trigonometry have especially need of such a precaution.

As an introduction to this practical part, there will be introduced at the close of this treatise an example of the calculation of each formula in an order the most concise, and most applicable to practice. In the complicated calculations of the practical application of trigonometry, it is useful to have forms of the process in blank, containing the order and denominations of the operations, and having a blank space sufficient for the insertion of the numbers. In this way the calculations may be reduced to an operation purely mechanical, in which no one of the necessary elements can possibly be omitted. This method has been long used in great geodetic works,and in navigation.

This treatise is then naturally divided into four parts.
1st Part. Analysis of the Trigonometric Functions.
2d ” Oblique angled Plane Trigonometry.
3d ” Spherical Trigonometry.
4th ” Examples of Calculation of the Formulæ of Plane and Spherical Trigonometry.

It only remains to give a few details in relation to some elementary principles made use of; and to such as are purely conventional, that it will become necessary to employ in this treatise.

Elementary Geometry teaches us that all the angles around any one point are together equal to four right angles; it follows that the circumference of a circle contains also four right angles.

The ordinary mode of expressing a right angle, is, $=\llcorner R$.

The division of the circumference of a circle is, from its very nature, conventional. Three different divisions have been used, at different times, and with different views; the most ancient of these enjoys the right derived from its priority of occupation. This is the division of the circumference into 360 equal parts, or *degrees;* each of these is divided into 60 equal parts, called *minutes,* (minutæ partes;) and these again into 60 parts, called *seconds,* (partes minutæ secundæ.) In the same manner we might proceed to obtain *thirds, fourths,* &c.; but instead of this, it is the custom at the present day, to represent the magnitudes of parts less than seconds, in the decimals of that denomination.

This division may therefore be represented in an algebraic form, (marking degrees by a small cypher above the numbers, minutes, by a single line, seconds, by two lines, &c.) as follows, viz:

$$\pi=360^\circ=4\,\llcorner\boldsymbol{R};\;\; 90^\circ=\llcorner\boldsymbol{R};\;\; 1^\circ=60';\;\; 1'=60''.$$

This furnishes the principle of the method of reduction, or transformation, of one denomination into another; and we might express the whole of the circumference in the following manner, viz:

$$\pi=3\,\llcorner\boldsymbol{R}+89^\circ\ 59'\ 60''.$$

The division of the fourth part of the circle or quadrant into 100°, with decimal subdivisions, has been several times attempted; in consequence of the usefulness such a division would possess, in all geodetic operations, when combined with the corresponding decimal metrical system.

The division of the quadrant into 96° has been employed by some of the best artists, in the graduation of great astronomical instruments. It is very advantageous in this process, because all the subdivisions, down to the single degree, may be obtained by the continual bisection of an arc, whose cord is equal to the radius of the circle; or in this division 64°, making in the ordinary division 60°.

The common division into 360° will be used in this treatise.

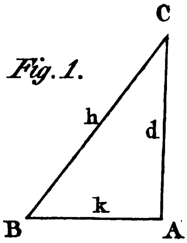
In order to show that the difference between two quantities is to be taken, in such a way that the result shall be always a positive quantity, which ever of the two be the greater, we shall use the sign $\backsim$, or an S lying horizontally.

The complement of an angle is that angle, which, when added to it, makes their sum a right angle. Thus the angle, b, has for its complement $90^\circ - b, = \llcorner R - b$.

The supplement of an angle is that angle, which, added to it, makes the sum equal to $2 \llcorner R = 180^\circ$. Thus the angle, b, has for its supplement, $2 \llcorner R - b, = 180^\circ - b$.

All other methods of notation, and the signs made use of, are derived from Algebra.

Fig. 1.



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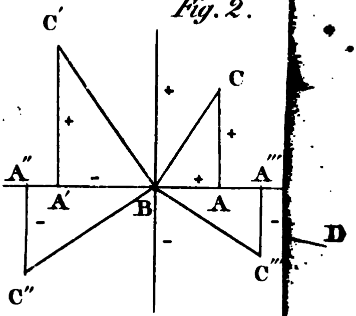

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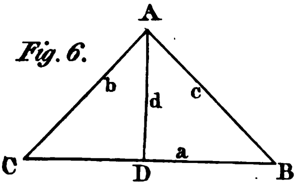

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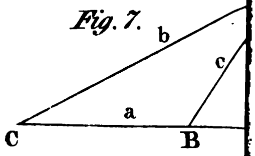

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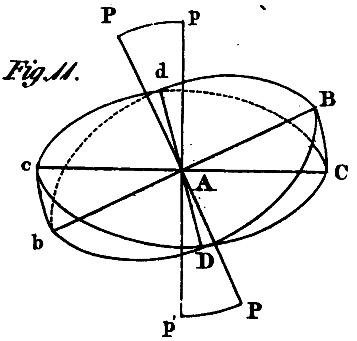

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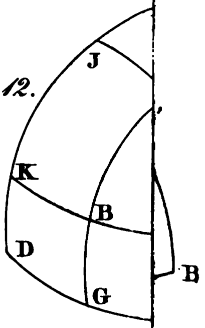

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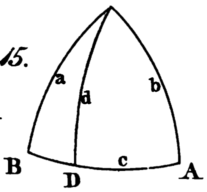

PART I.

ANALYSIS OF TRIGONOMETRIC FUNCTIONS.

CHAPTER I.

Deduction of the First elementary Formulæ.

§ 1. Analytic Trigonometry is one of the problems of Algebra applied to Geometry; it not only comprises all those solutions that are necessary to find the unknown parts of triangles from those which are known; but furnishes a series of formulæ and analytical expressions, that may be finally applied to Analysis in general; and which constitute a peculiar species of quantities called Trigonometric Functions. Considered in this point of view, it forms one of the most important branches of analytic mathematics.

§ 2. If the three angular points of a triangle be considered as lying in the same plane, in which, therefore, the lines which join these points are likewise situated, the triangle becomes the subject of the investigation of Plane Trigonometry. Elementary geometry makes us acquainted with the principles of equality and proportion that exist between them under certain relations of their several parts, and trigonometry employs these principles as the basis of its researches.

§ 3. If the angular points of the triangle be considered as not in the same plane, the triangle becomes, generally speaking, the subject of the investigation of Spherical Trigonometry, as it is referred to the curved surface generated by the revolution of the circumference of a circle around its diameter, or the surface of a sphere; its properties are derived from solid geometry; and it is the only curved surface that is considered in the elementary part of that branch of mathematics.

§ 4. It is evident that, in the extension of the subject, there may be a separate species of trigonometry for every possible variety of surface generated by the revolution of a re-entering curve. The equation of the radius of the curve would be an essential element of the resulting trigonometry; as, for instance, an ellipsoidic or spheroidic trigonometry. But this case requires a more complicated analysis; it is more detailed in its investigations, and consequently less general in its applications: it therefore cannot belong to elementary mathematics.

§ 5. The elementary trigonometric functions are the ratios that exist between the three sides forming a right angled plane triangle; or, in other words, the quotients that arise from dividing any one of them by either of the two others. There are not, therefore, necessarily more than three such functions, to which are added their inverse ratios. These several functions are known by names, whose origin and signification are of no importance; but it is the more important, that we fully and precisely understand their value, and mutual relations.

The combination of these ratios gives the whole of that multitude of trigonometric functions, that enable us to solve every question in trigonometry, and which are perpetually applied in analysis.

§ 6. Let, ABC, (figure 1) be a plane triangle, right angled at A; the sum, therefore, of the two other angles, $B+C=\llcorner R=90^\circ$. They are, consequently, each the difference between the other and a right angle. This relation of these two angles being the complement, as has been previously stated, we have, according to the division of the circle into 360°, $B=90^\circ-C$; and $C=90^\circ-B$.

The theorem of elementary geometry known by the name of Pythagoras (Euclid, Book I. prop. 47) gives the following relations:

$$BC^2 = AB^2 + AC^2, \text{ whence}$$
$$AB^2 = BC^2 - AC^2, \text{ and}$$
$$AC^2 = BC^2 - AB^2.$$

To simplify these expressions, let BC$=h$, AB$=k$, AC$=d$, and we have

$$h^2 = k^2 + d^2$$
$$k^2 = h^2 - d^2$$
$$d^2 = h^2 - k^2$$

which determine the relations between the sides of a right angled plane triangle, in terms of their squares.

§ 7. To these properties of a right angled triangle, given in elementary geometry, trigonometry adds the expressions that denote the *ratios* of the several sides; or rather, it gives to each of these *ratios* a specific name, as follows, viz:

The ratio,	or the quotient,	is called	A
$AC : BC$,	$\frac{d}{h}$	$=$ sine $B =$ cosine $(90^\circ - B) =$ cosine C	1
$AB : BC$,	$\frac{k}{h}$	$=$ cosine $B =$ sine $(90^\circ - B) =$ sine C	2
$CA : BA$,	$\frac{d}{k}$	$=$ tangent $B =$ cotangent $(90^\circ - B) =$ cotangent C	3
$BA : CA$,	$\frac{k}{d}$	$=$ cotangent $B =$ tangent $(90^\circ - B) =$ tangent C	4
$BC : AB$,	$\frac{h}{k}$	$=$ secant $B =$ cosecant $(90^\circ - B) =$ cosecant C	5
$BC : AC$,	$\frac{h}{d}$	$=$ cosecant $B =$ secant $(90^\circ - B) =$ secant C	6

It is evident from inspection, that the prefix, co, before the names sine, tangent, secant, show that the relations of the quantities are the same when they are referred to the complementary angle, as when with their simple names, they are considered in relation to the angle itself.

It is also evident that the three last ratios are the inverse of the three first. They are consequently much less used than the three first, particularly the two last: these are, indeed, at present entirely neglected, as well as the terms, versed sine, and co-versed sine; having all become useless, in consequence of the great simplification that has taken place in trigonometric formulæ.

§ 8. Combining the primitive formulæ thus found, or determined, by their multiplication and division, and comparing the results with the simple formulæ, or definitions, to which the products or quotients are equal, we obtain a series of functions, or formulæ, that constitute what may be called the multiplication table of analytic trigonometry. Thus:

B By the multiplication of

1 A No 1 into No 6 or $\frac{d}{h},\frac{h}{d}=\text{sine}\,B\ \text{cosec}\,B=1$

2 2 5 $\frac{k}{h},\frac{h}{k}=\cos B\ \sec B=1$

3 3 4 $\frac{d}{k},\frac{k}{d}=\tan B.\ \cot B=1$

4 1 4 $\frac{d}{h},\frac{k}{d}=\frac{k}{h}=\text{sine}\,B\ \cot B=\cos B$

5 2 3 $\frac{k}{h},\frac{d}{k}=\frac{d}{h}=\text{cosine}\,B\ \tan B=\text{sine}\,B$

6 1 5 $\frac{d}{h},\frac{h}{k}=\frac{d}{k}=\text{sine}\,B\ \sec B=\tan B$

7 2 6 $\frac{k}{h},\frac{h}{d}=\frac{k}{d}=\cos B\ \text{cosec}\,B=\cot B$

By the division of

8 *H* 1 by No 2 or $\frac{d}{h}:\frac{k}{h}=\frac{d}{k}=\frac{\text{sine}\,B}{\cos B}=\tan B$

9 2 1 $\frac{k}{h}:\frac{d}{h}=\frac{k}{d}=\frac{\cos B}{\text{sine}\,B}=\cot B$

10 1 3 $\frac{d}{h}:\frac{d}{k}=\frac{k}{h}=\frac{\text{sine}\,B}{\tan B}=\cos B$

No 3 by No 1 or $\frac{d}{k}:\frac{d}{h}=\frac{h}{k}=\frac{\tan B}{\text{sine } B}=\sec B$ 11

2 4 $\frac{k}{h}:\frac{k}{d}=\frac{d}{h}=\frac{\cos B}{\cot B}=\text{sine } B$ 12

4 2 $\frac{k}{d}:\frac{k}{h}=\frac{h}{d}=\frac{\cot B}{\cos B}=\text{cosec } B$ 13

3 5 $\frac{d}{k}:\frac{h}{k}=\frac{d}{h}=\frac{\tan B}{\sec B}=\text{sine } B$ 14

5 3 $\frac{h}{k}:\frac{d}{k}=\frac{h}{d}=\frac{\sec B}{\tan B}=\text{cosec } B$ 15

4 6 $\frac{k}{d}:\frac{h}{d}=\frac{k}{h}=\frac{\cot B}{\text{cosec } B}=\cos B$ 16

6 4 $\frac{h}{d}:\frac{k}{d}=\frac{h}{k}=\frac{\text{cosec } B}{\cot B}=\sec B$ 17

5 6 $\frac{h}{k}:\frac{h}{d}=\frac{d}{k}=\frac{\sec B}{\text{cosec } B}=\tan B$ 18

6 5 $\frac{h}{d}:\frac{h}{k}=\frac{k}{d}=\frac{\text{cosec} B}{\sec B}=\cot B$ 19

If the combinations producing squares were admitted into this table, it would become more extensive, but it is not considered proper to introduce them here, as they may be considered with more propriety as consequences.

It will also be observed that some of the above results are already repetitions, for they may be considered as algebraically contained in preceding ones; but this, being exactly analogous to what occurs in the common multiplication table, has been admitted, in the same way as, in a complete multiplication table, two equal products, say, for instance, 3 times 3, and 4 times 3, are introduced, to accustom the beginner to their equality.

§ 9. If we apply the three expressions deduced from the 47th Prop. of Euclid, Book I. given in § 6, viz :

$$h^2 = d^2 + k^2;\ d^2 = h^2 - k^2;\ k^2 = h^2 - d^2;$$

to those found in the series, A, making use of the expressions that have the same denominator, and reducing the numerators, resulting from the addition or subtraction of their squares, we obtain a new series of formulæ, that give the relations of the squares of the several functions; viz :

C The sum of the squares of

1 1 and 2, or $\frac{d^2}{h^2} + \frac{k^2}{h^2} = \frac{d^2+k^2}{h^2} = \frac{h^2}{h^2} = 1 = \sin^2 B + \cos^2 B$

The difference of the squares of

2 5 and 3, or $\frac{h^2}{k^2} - \frac{h^2}{k^2} = \frac{h^2-d^2}{k^2} = \frac{k^2}{k^2} = 1 = \sec^2 B - \tan^2 B$

3 6 and 4, $\frac{h^2}{d^2} - \frac{k^2}{d^2} = \frac{h^2-k^2}{d^2} = \frac{d^2}{d^2} = 1 = \operatorname{cosec}^2 B - \cot^2 B$

From these equations are obtained, by simply transposing the terms, the following, which are of very frequent use in trigonometric calculations.

4 $\sin^2 B = 1 - \cos^2 B$

5 $\cos^2 B = 1 - \sin^2 B$

6 $\sec^2 B = 1 + \tan^2 B$

7 $\operatorname{cosec}^2 B = 1 + \cot^2 B$

8 $\tan^2 B = \sec^2 B - 1$

9 $\cot^2 B = \operatorname{cosec}^2 B - 1$

By equalizing the first three results, it is also evident that

10 $\sin^2 B + \cos^2 B = \sec^2 B - \tan^2 B = \operatorname{cosec}^2 B - \cot^2 B = 1$

and

11 $\sec^2 B - \operatorname{cosec}^2 B = \tan^2 B - \cot^2 B$

with their several consequences.

§ 10. Combining the two series of formulæ, B, and, C, by simply substituting the roots taken from, C, in the formulæ of B, we obtain another series of formulæ, of frequent use in the application of logarithms to trigonometrical calculations, and in the integral calculus.

From what has been observed, and has been already shown, it is sufficient to give these for sines, cosines, and tangents; for which the following values will be successively obtained.

By B	and the substitution from		D
No. 5	C No. 6	$\text{sine } B = \cos B\,(\sec^2 B - 1)^{\frac{1}{2}}$	1
1	7	$= \dfrac{1}{(1 + \cot^2 B)^{\frac{1}{2}}}$	2
14	6	$= \dfrac{\tan B}{(1 + \tan^2 B)^{\frac{1}{2}}}$	3
12	7	$= \dfrac{\cos B}{(\text{cosec}^2 B - 1)^{\frac{1}{2}}}$	4
5	5 and 6	$= (1 - \sin^2 B)^{\frac{1}{2}}\,(\sec^2 B - 1)^{\frac{1}{2}}$	5
14	2 and 6	$= \dfrac{(\sec B - 1)^{\frac{1}{2}}}{(1 + \tan^2 B)^{\frac{1}{2}}}$	6
12	5 and 6	$= \dfrac{(1 - \sin^2 B)^{\frac{1}{2}}}{(\text{cosec}^3 B - 1)^{\frac{1}{2}}}$	7
By B	substituting from		
No. 4	C 9	$\text{cosine } B = \text{sine } B\,(\text{cosec}^2 B - 1)^{\frac{1}{2}}$	8
2	6	$= \dfrac{1}{(1 + \tan^2 B)^{\frac{1}{2}}}$	9
10	6	$= \dfrac{\text{sine } B}{(\sec^2 B - 1)^{\frac{1}{2}}}$	10
16	9	$= \dfrac{(\text{cosec}^2 B - 1)^{\frac{1}{2}}}{\text{cosec } B}$	11
4	4 and 9	$= (1 - \sin^2 B)^{\frac{1}{2}}\,(\text{cosec}^2 B - 1)^{\frac{1}{2}}$	12

	By B	substituting from	
13	10	C 4 and 6	$=\dfrac{(1-\cos^2 B)^{\frac{1}{2}}}{(\sec^2 B-1)^{\frac{1}{2}}}$
14	16	7 and 9	$=\dfrac{(\operatorname{cosec}^2 B-1)^{\frac{1}{2}}}{(1+\cot^2 B)^{\frac{1}{2}}}=\dfrac{\cot B}{(1+\cot^2 B)^{\frac{1}{2}}}$
15	No. 3	No. 9	$\tan B=\dfrac{1}{(\operatorname{cosec}^2 B-1)^{\frac{1}{2}}}$
16	6	8	$=\sin B\,(1+\tan\ B)^{\frac{1}{2}}$
17	8	5	$=\dfrac{\sin B}{(1-\sin^2 B)^{\frac{1}{2}}}$
18	18	6	$=\dfrac{(1+\tan^2 B)^{\frac{1}{2}}}{\operatorname{cosec}^2 B}$
19	6	4 and 6	$=(1-\cos^2 B)^{\frac{1}{2}}\,(1+\tan^2 B)^{\frac{1}{2}}$
20	8	4 and 5	$=\dfrac{(1-\cos^2 B)^{\frac{1}{2}}}{(1-\sin^2 B)^{\frac{1}{2}}}$
21	18	6 and 7	$=\dfrac{(1+\tan^2 B)^{\frac{1}{2}}}{(1+\cot^2 B)^{\frac{1}{2}}}$

It is evident, that the formulæ for the sine will give those for the cosecant, by merely changing the denominators into numerators, and the numerators into denominators; or, in other words, by expressing the inverse ratio of the sine. In like manner, by performing a similar operation, the values of the cosine will give those for the secant, and those of the tangent the values of the cotangent.

CHAPTER II.

Solution of right angled plane Triangles; Values and Algebraic signs of certain Trigonometric Functions.

§ 11. The formulæ of the preceding chapter are evidently true whatever be the magnitude of the angle *B*, and the ratio for a given angle being given by any one of the functions of the series A, the determination of the value of any one of the lines, *h*, *d*, *k*, will, it is manifest, give the value of the two others.

From this it results, that these formulæ contain the solution of every possible case of a right angled plane triangle. It will suffice for this purpose to make choice of that trigonometric function, in the equation of which, the known quantity is in the denominator of the fraction expressing it, and the unknown quantity in the numerator; and to multiply the trigonometric function of the corresponding angle by the denominator of the fraction; to obtain for result the unknown quantity which is represented by the numerator. For, every ratio being a fraction, or quotient, representing the relative value of two quantities, in which the denominator points out the value of each of the parts; the multiplication of the quotient by the absolute value of all the parts, must present in the result the absolute value of the numerator This principle is evident from the manner in which the trigonometric functions have been deduced, and is general; it would therefore be useless to enter into any detail.

§ 12. In correspondence with the general principle just stated, the numerical values of these several quotients have been calculated, for all angles from, 0°, to 90°, on the supposition that the value of the denominator is constantly unity; they are therefore directly applicable by means of the rule just given.

As in a right angled triangle one of the acute angles is always the complement of the other, it follows: that when either of them is half a right angle or = 45°, the lines, *k*, and, *d*, becoming equal, their trigonometric functions of corresponding denomination are also equal, that is to say:

sine = cosine
tangent = cotangent
secant = cosecant

And as, on the angle becoming greater than 45°, the complementary angle takes, in succession, every value of the primitive angle, in an inverted order, it follows: that in an angle between 45° and 90°, the simple change of any one of the above denominations of functions into its corresponding one will give the function sought. For this reason it is only necessary to calculate the value of the sines, cosines, tangents, and cotangents, from 0° to 45°, in order to obtain every other value that is necessary.

§ 13 Let it now be supposed, that any line, *BC*=*h*, (figure 2) take successively all possible positions around the point, *B*, so as to form in relation to a fixed line, *BA*, successively, all the angles from, 0°, to, 360°, in which last position it will again coincide with, 0°, and if we conceive a perpendicular to fall in any position of the line from a point, *C*, taken at any distance whatsoever from the point, *B*, upon the line, *BA*, produced indefinitely on either side of the point, *B*; and if, according to the constant supposition in geometry, we assign to this line, and to the perpendicular, the proper algebraic signs, to show their direction in relation to the point, *B*, giving the sign, +, to these positions of the lines, *d*, and *k*, that correspond in their direction with their primitive position, and the sign, —, where they are in an opposite direction; there will result all the variations of value, in quantity and in sign, that these elementary functions can possibly assume.

In order to show this more clearly: Let, *BC*, *BC'*, *BC''*, *BC'''*, (figure 2) be several successive positions of this line, in the four right angles, which are contained around the point, *B*, the lines, *d*, and, *k*, will take the signs assigned to them in the figure, and the signs of the fundamental trigonometric functions contained in the series A, will always be determined, upon the general and simple principle, that serves to determine the signs in algebra; that is to say, that like signs produce, +, and unlike ones, —. If therefore we compare with the formulæ, the lines, *k*, and, *d*, of the figure, in regard to their respective positions, it will be found: that, supposing all the functions within the first right angle to be positive, we shall have in the

2d right angle the,	sines, and, cosecants, +,	the other functions, —,
3d	tangent, and, cotangent, +,	,—,
4th	cosine, and, secant, +,	,—,

§ 14 In the passage of, *h*, from one quadrant to another, as well as in its first position, the lines, *d*, and, *k*, become alternately equal to, 0, and to, *h*, itself. In these cases they evidently acquire their least and greatest possible values.

If, therefore, we suppose, $h=1$, and use, π, to represent the entire circumference of a circle, the point, 0°, or the origin of the angles, will be represented by, 0π, the first quadrant or right angle, by, $\frac{1}{4}\pi$, and so on. Hence the values of the trigonometric functions in these four principal positions, when expressed in terms of, π, will assume the following values, viz: E

For, 0π, we shall have, $d = 0$; and, $k = 1$ 1

which gives $$\frac{d}{h} = \frac{0}{1} = \sin 0\pi = 0$$

$$\frac{k}{h} = \frac{1}{1} = \cos 0\pi = 1$$

$$\frac{d}{k} = \frac{0}{1} = \tan 0\pi = 0$$

$$\frac{k}{d} = \frac{1}{0} = \cot 0\pi = \text{infinit}$$

$$\frac{h}{d} = \frac{1}{0} = \text{cosec } 0\pi = \text{infinit}$$

$$\frac{h}{k} = \frac{1}{1} = \sec 0\pi = 1$$

2 For, $\frac{1}{4}\pi$, we have, $d = 1$, and, $k = 0$,

giving $$\frac{d}{h} = \frac{1}{1} = \sin \tfrac{1}{4}\pi = 1$$

$$\frac{k}{h} = \frac{0}{1} = \cos \tfrac{1}{4}\pi = 0$$

$$\frac{d}{k} = \frac{1}{0} = \tan \tfrac{1}{4}\pi = \text{infinit}$$

$$\frac{k}{d} = \frac{0}{1} = \cot \tfrac{1}{4}\pi = 0$$

$$\frac{h}{d} = \frac{1}{1} = \text{cosec } \tfrac{1}{4}\pi = 1$$

$$\frac{h}{k} = \frac{1}{0} = \sec \tfrac{1}{4}\pi = \text{infinit}$$

3 For, $\frac{2}{4}\pi$, we have, $d = 0$; and, $(k = -1,)$

giving $$\frac{d}{h} = \frac{0}{1} = \text{sine } \tfrac{2}{4}\pi = 0$$

$$\frac{-k}{h} = \frac{-1}{1} = \cos \tfrac{2}{4}\pi = -1$$

$$\frac{d}{-k} = \frac{0}{-1} = \tan \tfrac{2}{4}\pi = -0$$

$$\frac{-k}{d} = \frac{-1}{0} = \cot \tfrac{2}{4}\pi = -\text{ infinit}$$

$$\frac{h}{d} = \frac{1}{0} = \text{cosec } \tfrac{2}{4}\pi = +\text{ infinit}$$

$$\frac{h}{-k} = \frac{1}{-1} = \sec \tfrac{2}{4}\pi = -1$$

For $\frac{3}{4}\pi$ we have, $d = -1$, and, $k = 0$, 4

giving $$\frac{-d}{h} = \frac{-1}{1} = \sin \tfrac{3}{4}\pi = -1$$

$$\frac{k}{h} = \frac{0}{1} = \cos \tfrac{3}{4}\pi = 0$$

$$\frac{-d}{k} = \frac{-1}{0} = \tan \tfrac{3}{4}\pi = -\text{ infinit}$$

$$\frac{k}{-d} = \frac{0}{-1} = \cot \tfrac{3}{4}\pi = -0$$

$$\frac{h}{-d} = \frac{1}{-1} = \operatorname{cosec} \tfrac{3}{4}\pi = -1$$

$$\frac{h}{k} = \frac{1}{0} = \sec \tfrac{3}{4}\pi = \text{infinit}$$ (*)

§ 15. It is evident, from what has been said in the two sections immediately preceding, that all the elementary trigonometric functions may be represented in a circle, whose radius is, $h = 1$; and that they will always form proper or improper fractions of this unit, from 0, to infinity.

In fig. 3, let, B, be the centre of the circle, whose radius, $h = 1$, BA, and, Ba, two radii at right angles to each other, that contain the first quadrant; the points, C, C', &c. the successive intersections of, h, with the circumference in the four quadrants; the perpendiculars let fall from the points C, C',

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(*) The expression $\frac{1}{0}$ which is here seen to result from the division of the different lines gives the best idea of what is called infinity; for it appears as a ratio (or relative quantity) such as it would exceed the power of any number to express. The sign commonly used for it in analysis is, ∞, or an 8 placed horizontally.

&c. upon the radius BA, produced upon the other side of B, will represent, both in magnitude and algebraic sign, in relation to, $h = 1$, the sines of the angles ABC, ABC', &c. while the parts of the line BA, intercepted between the perpendiculars and the point, B, will represent the several cosines of the same angles.

Draw from C, a line parallel to AB, until it intersect the line Ba, the lines, Cl, and Bl, are equal to Bg, and Cg, each to each; whence it is manifest, that, as the angle aBC, is the complement of ABC, we have

$$\text{sine } ABC = \cos aBC$$
$$\cos ABC = \sin aBC$$

In the same manner, if we draw from the points A, and a, perpendiculars, upon BA, and Ba, produced in either direction from A, and a, the line BC, BC', &c. produced on either side of B, will cut these perpendiculars in points, such as c, c', e, e', and Ac, will represent the tangent; ac' the cotangent, Bc the secant, Bc' the cosecant of the angle ABC; and in these functions of the angle, ABC, the same relation takes place with respect to the exchange of the denominations of these functions, that we have seen to occur in regard to the sines and cosines, &c. of this angle and its complement aBC, for we have

$$\tan ABC = \cot aBC$$
$$\cot ABC = \tan aBC$$
$$\sec ABC = \text{cosec } aBC$$
$$\text{cosec } ABC = \sec aBC$$

The figure shows in what manner the signs of these quantities are affected in the four quadrants; attention being paid to the principle, that A, and a, are always the points from which the tangents are considered to be drawn in either direction, in which they can cut the produced radius, or h. We must be careful here to avoid falling into the error of supposing a change of sign in h; the radius of a circle can never be any thing but a positive quantity; it is only the effect of its position upon the perpendiculars, considered in relation to the directions of *BA*, and *Ba*, which depend for their sign upon the position of *h*, in the several quadrants, that can be affected by different signs; for in nature, and consequently in mathematics, every efficient cause is positive, while it is only its effect, in regard to a required result, that may become negative.

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CHAPTER III.

Fundamental Trigonometric Functions of the Sum, and Difference of two Angles.

§ 16. *Problem.* To find the sine and cosine of the sum and difference of two angles, their respective sines and cosines being given.

Let *DBC*, and, *ABC*, (in figures 4, and 5,) be the two angles, placed upon the common line, *BC*, in such a manner that the angle, *ABD*, may represent their sum, (in figure 4,) or difference, (in figure 5,) when, *ABD*, represents the sum, the two angles will then each fall without the other; when it represents their difference, the less will be included in the greater; it is required to find the sine and the cosine of their sum or difference, or of the angle *ABD*.

Construction. Through any point *E*, in the line *BC*, that is common to the two angles, draw a perpendicular *FG*, cutting the two other lines *BA*, and *BD*, in the points *F*, and G. From the point, *F* where this perpendicular cuts the line *BA*, which marks the sum or difference of these angles, let fall the perpendicular *FH*, upon the third line, *BD*.

Using the same denominations as in the primitive formulæ of series A, we make: $BG=h$; $BF=h'$; $BE=k$; $EG=d$; $EF=d'$, and calling the angle $CBD=a$; the angle $CBA=b$; and the perpendicular $FH=y$; and $BH=x$.

The $FG=d\pm d'$, will follow, with the sign +, for the sum, and —, for the difference, of the two angles *a*, and *b*, and we shall have the quotient or ratio:

$$\frac{y}{h'} = \sin\,(a \pm b)\,; \text{ and } \frac{x}{h'} = \cos\,(a \pm b)$$

Solution. **The triangles *FGH,* and *BGE,* are similar, being right angled at *H,* and *E,* and having the angle *G,* common to the two triangles ; wherefore**

(By Euclid, B. 6. Prop. 4.) $h : k = d \pm d' : y$

$$y = \frac{k\,d \pm k\,d'}{h}$$

Dividing by h',

$$\frac{y}{h'} = \frac{k\,d \pm k\,d'}{h\,h'}$$

$$= \frac{k\,d}{h'\,h} \pm \frac{k\,d'}{h\,h'}$$

$$= \frac{k}{h'} . \frac{d}{h} \pm \frac{k}{h} . \frac{d'}{h'}$$

Substituting the values of these several quotients, according
F to the principles of the series A, we have

1 $$\sin\,(a \pm b) = \sin a \cos b \pm \cos a \sin b$$

For the cosine we have

(Euc. B 1. Prop. 47.) $y^2 = (d \pm d')^2 - (h - x)^2 = (h')^2 - x^2$

or $(d \pm d')^2 - h^2 + 2\,hx - x^2 = (h')^2 - x^2$

and $(d \pm d')^2 - h^2 + 2\,hx = (h')^2$

therefore $2\,hx = (h')^2 + h^2 - (d \pm d')^2$

Substituting for $h^2 = k^2 \pm d^2$; and $(h')^2 = k^2 + (d')^2$; and dividing by $2hh'$,

$$\frac{x}{h'} = \frac{2\,k^2 + (d')^2 + d^2 - (d \pm d)^2}{2hh'}$$

Squaring $(d \pm d')$ $$= \frac{2k^2 + (d')^2 + d^2 - (d') - d^2 \pm 2dd'}{2hh'}$$

By compensation $$= \frac{k\,k \mp d\,d'}{h\,h'}$$

$$\frac{x}{h'} = \frac{k}{h} \cdot \frac{k}{h'} \mp \frac{d}{h} \cdot \frac{d'}{h'}$$

Substituting for these quotients their values, according to the series A, we have

$$\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b \qquad 2$$

It will be here seen, that in the result the algebraic signs of the last formulæ, are of the contrary nature to that they possess in the expression representing the sum or difference of the two angles; while in the case of the sines they have the same nature, as in the expression of the compound angle. This might also have been anticipated from the simple knowledge of the fact, that the cosine diminishes with the increase of the angle; for in every greater angle the line k, will be less, than in a less angle; while the perpendiculars increase with the increase of the angle.

§ 17. In order to find the tangent, cotangent, secant and cosecant, of the sum, or difference, of two angles; we must treat these formulæ, 1, and 2, in the same way as the simple formulæ of the series A, when those of the series B, were investigated; and then simplify them by means of these same formulæ, in conformity with what was at first said in relation to them, that they constitute the multiplication table of trigonometry, and thus furnish the means of reduction. We shall then have, (analogous to B, No. 8,)

$$\tan(a \pm b) = \frac{\sin(a \pm b)}{\cos(a \pm b)} = \frac{\sin a \cos b \pm \cos a \sin b}{\cos a \cos b \mp \sin a \sin b} \qquad 3$$

dividing this last expression in numerator and denominator, successively by the four factors contained in it, and substituting, for the resulting values, the corresponding tangents and cotangents, according to the formulæ of the series, B, we obtain in succession the following formulæ, viz.

Dividing by $\sin a \cos b$; $\tan(a \pm b) = \dfrac{1 \pm \tan b \cot a}{\cot a \mp \tan b}$ 4

5 $$\sin b \cos a = \frac{\tan a \cot b \pm 1}{\cot b \mp \tan a}$$

6 $$\cos a \cos b = \frac{\tan a \pm \tan b}{1 \mp \tan a \tan b}$$

7 $$\sin a \sin b = \frac{\cot b \pm \cot a}{\cot a \cot b \mp 1}$$

For the value of the cotangent is obtained, analagous to B, 9.

8 $$\cot (a \pm b) = \frac{\cos (a \pm b)}{\sin (a \pm b)} = \frac{\cos a \cos b \mp \sin a \sin b}{\sin a \cos b \pm \cos a \sin b}$$

A process analogous to the preceding gives in succession the following formulæ :

9 Dividing by, $\sin a \cos b$, $$\cot (a \pm b) = \frac{\cot a \mp \tan b}{1 \pm \tan b \cot a}$$

10 $$\sin b \cos a = \frac{\cot b \mp \tan a}{\tan a \cot b \pm 1}$$

11 $$\cos a \cos b = \frac{1 \mp \tan a \tan b}{\tan a \pm \tan b}$$

12 $$\sin a \sin b = \frac{\cot a \cot b \mp 1}{\cot b \pm \cot a}$$

It may be easily seen that these formulæ for the cotangent are the inverse of those for the tangent, as might be expected from their analogy to A, No. 3, and 4.

In the same manner as before, we obtain

13 $$\sec (a \pm b) = \frac{1}{\cos (a \pm b)} = \frac{1}{\cos a \cos b \mp \sin a \sin b}$$

Dividing still in this case by the same four factors, employed in the case of the tangent, and substituting the, secants, and, cosecants, for their equals, $\frac{1}{\cos}$ & $\frac{1}{\sin}$, in conformity with the expressions of the series, B, we obtain the four following results, viz.

$$\sec(a \pm b) = \frac{\operatorname{cosec} a \sec b}{\cot a \mp \tan b} \tag{14}$$

$$\sec(a \pm b) = \frac{\sec a \operatorname{cosec} b}{\cot b \mp \tan a} \tag{15}$$

$$= \frac{\sec a \sec b}{1 \mp \tan a \tan b} \tag{16}$$

$$= \frac{\operatorname{cosec} a \operatorname{cosec} b}{\cot a \cot b \mp 1} \tag{17}$$

Substituting for the secants their values in terms of the tangents, taken from the radical expressions of series C, as has been done for series D, these formulæ undergo the following transformations, which may easily be followed without being detailed :

$$\sec(a \pm b) = \frac{(1 + \cot^2 a)^{\frac{1}{2}} (1 + \tan^2 b)^{\frac{1}{2}}}{\cot a \mp \tan b} \tag{18}$$

$$= \frac{(1 + \tan^2 a)^{\frac{1}{2}} (1 + \cot^2 b)^{\frac{1}{2}}}{\cot b \mp \tan a} \tag{19}$$

$$= \frac{(1 + \tan^2 a)^{\frac{1}{2}} (1 + \tan^2 b)^{\frac{1}{2}}}{1 \mp \tan a \tan b} \tag{20}$$

$$= \frac{(1 + \cot^2 a)^{\frac{1}{2}} (1 + \cot^2 b)^{\frac{1}{2}}}{\cot a \cot b \mp 1} \tag{21}$$

Applying a process exactly analogous to the expressions of the value of the cosecant, we obtain successively the following formulæ, which are analogous to the preceding ones :

$$\operatorname{cosec}(a \pm b) = \frac{1}{\sin(a \pm b)} = \frac{1}{\sin a \cos b \pm \cos a \sin b} \tag{22}$$

$$= \frac{\operatorname{cosec} a \sec b}{1 \pm \cot a \tan b} \tag{23}$$

24 $$\text{cosec } (a \pm b = \frac{\sec a \text{ cosec } b}{\tan a \cot b \pm 1}$$

25 $$= \frac{\text{cosec } a \text{ cosec } b}{\cot b \pm \cot a}$$

26 $$= \frac{\sec a \sec b}{\tan a \pm \tan b}$$

27 $$= \frac{(1 + \cot^2 a)^{\frac{1}{2}} (1 + \tan^2 b)^{\frac{1}{2}}}{1 \pm \cot a \tan b}$$

28 $$= \frac{(1 + \tan^2 a)^{\frac{1}{2}} (1 + \cot^2 b)^{\frac{1}{2}}}{\tan a \cot b \pm 1}$$

29 $$= \frac{(1 + \cot^2 a)^{\frac{1}{2}} (1 + \cot^2 b)^{\frac{1}{2}}}{\cot b \pm \cot a}$$

30 $$= \frac{(1 + \tan^2 a)^{\frac{1}{2}} (1 + \tan^2 b)^{\frac{1}{2}}}{\tan a \pm \tan b}$$

It is evident, that, if in these formulæ for secant, and cosecant, we should change the numerators into denominators, and the denominators into numerators, we should obtain expressions for the sine, and cosine; in their inverse application all these formulæ are naturally reductions of compound expressions to the simple expressions of a compound angle; if therefore we meet with such formulæ as the above in the course of a calculation, we have the means furnished us of rendering them much more simple.

CHAPTER IV.

Combinations of the Formulæ of Simple Angles.

§ 18. THE use which we have made, in the last chapter of the formulæ of the series B, has given an instance of the value of the research of the combinations of trigonometric functions, as applicable to the reduction of complicated formulæ, as well as in obtaining expressions appropriate to the data that may present themselves in calculation.

As it is evident, that these combinations ought to be the result of the application of one or the other of the four rules of arithmetic, the investigation will be here made by this simple method.

It is clear, that these combinations must be very numerous; we shall therefore, in this place, rather point out the road, that leads to their discovery, than enter into a detail of all the possible combinations.

One of the frequent uses that is made of these formulæ, consists in changing an addition or subtraction into a multiplication, (in order to enable us to make use of logarithms,) and conversely. We shall therefore devote ourselves, principally, to formulæ that have properties of this sort. It will be easy, by a slight attention to the general method, to reach any other form that may be desired in any particular case.

§ 19. The simple addition and subtraction of the formula B, No. 8, applied to two angles, a, and b, assuming, $a > b$, will give

G

$$\tan a \pm \tan b = \frac{\sin a}{\sin b} \pm \frac{\sin b}{\cos b} = \frac{\sin a \cos b \pm \cos a \sin b}{\cos a \cos b} \qquad 1$$

$$= \frac{\sin (a \pm b)}{\cos a \cos b}$$

2 Dividing by, $\sin a \cos b$; $\tan a \pm \tan b = \dfrac{1 \pm \cot a \tan b}{\cot a}$

3 $\cos a \sin b$; $= \dfrac{\tan a \cot b \pm 1}{\cot b}$

4 $\sin a \sin b$; $= \dfrac{\cot b \pm \cot a}{\cot a \cot b}$

From B, No. 9, treated in the same manner, we obtain

5 $\cot b \pm \cot a = \dfrac{\cos b}{\sin b} \pm \dfrac{\cos a}{\sin a} = \dfrac{\sin a \cos b \pm \cos a \sin b}{\sin a \sin b} = \dfrac{\sin (a \pm b)}{\sin a \sin b}$

6 Dividing by, $\sin a \cos b$; $\cot b \pm \cot a = \dfrac{1 \pm \cot a \tan b}{\tan b}$

7 $\cos a \sin b$; $= \dfrac{\tan a \cot b \pm 1}{\tan a}$

8 $\cos a \cos b$; $= \dfrac{\tan a \pm \tan b}{\tan a \tan b}$

From the combination of B, No. 8 & 9, applied to different angles, we obtain:

9 $\cot a \pm \tan b = \dfrac{\cos a}{\sin a} \pm \dfrac{\sin b}{\cos b} = \dfrac{\cos a \cos b \pm \sin a \sin b}{\sin a \cos b} = \dfrac{\cos (a \mp b)}{\sin a \cos b}$

10 Dividing by, $\cos a \cos b$; $\cot a \pm \tan b = \dfrac{1 \pm \tan a \tan b}{\tan a}$

11 $\sin a \sin b$; $= \dfrac{\cot a \cot b \pm 1}{\cot b}$

12 $\cos a \sin b$; $= \dfrac{\cot b \pm \tan a}{\tan a \cot b}$

and

13 $\cot b \pm \tan a = \dfrac{\cos b}{\sin b} \pm \dfrac{\sin a}{\cos a} = \dfrac{\cos a \cos b \pm \sin a \sin b}{\sin b \cos a} = \dfrac{\cos (a \mp b)}{\sin b \cos a}$

Dividing by, $\sin a \cos b$; $$\cot b \pm \tan a = \frac{1 \pm \tan a \tan b}{\tan a} \tag{14}$$

$\sin a \sin b$; $$= \frac{\cot a \cot b \pm 1}{\cot a} \tag{15}$$

$\cos b \sin a$; $$= \frac{\cot a \pm \tan b}{\cot a \tan b} \tag{16}$$

The formulæ, No. 2, 3, 4; 6, 7, 8; 10, 11, 12; 14, 15, 16; might evidently have been obtained, with equal ease, by the simple multiplication or division of the sums indicated by tangent a, tangent b, or their products; but as they naturally follow, from the method that has been employed previously, and since, in this way, the different values of the sums sought are collated, it seems to be more in conformity with systematic arrangement, to present them in the way they occur above.

§ 20. The several combinations of the formulæ, 8 & 9, of series B, by means of multiplication and division, are, as is clear, contained in those which precede. In effect we have, by comparing the formulæ No. 5 & 8; No. 4 & 9; No. 9 & 12; No. 13 & 16, the following:

H

$$\tan a \tan b = \frac{\tan a}{\cot b} = \frac{\sin (a \pm b)}{\sin a \sin b (\tan a \pm \tan b)} \tag{1}$$

$$\cot a \cot b = \frac{\cot a}{\tan b} = \frac{\sin (a \pm b)}{\cos a \cos b (\cot b \pm \cot a)} \tag{2}$$

$$\tan a \cot b = \frac{\tan a}{\tan b} = \frac{\cos (a \mp b)}{\sin a \cos b (\cot b \pm \tan a)} \tag{3}$$

$$\cot a \tan b = \frac{\cot a}{\cot b} = \frac{\cos (a \mp b)}{\sin b \cos a (\cot a \pm \tan b)} \tag{4}$$

§ 21. The formulæ G, No. 1, 5, 9, & 13, are, as is evident, of such a nature as to change an addition or subtraction into a multiplication or division; they also serve, inversely, in the construction of tables to find the tangents, by means of the sines and cosines. In like manner we obtain, by comparing: G, No. 1, 2, and 3, and No. 9, 10, and 11, the following formulæ that will be of use.

1 $$1 \pm \cot a \tan b = \frac{\sin (a \pm b) \cot a}{\cos a \cos b} = \frac{\sin (a \pm b)}{\sin a \cos b}$$

2 $$\tan a \cot b \pm 1 = \frac{\sin (a \pm b) \cot b}{\cos a \cos b} = \frac{\sin (a \pm b)}{\cos a \sin b}$$

3 $$1 \pm \tan a \tan b = \frac{\cos (a \mp b) \tan a}{\sin a \cos b} = \frac{\cos (a \mp b)}{\cos a \cos b}$$

4 $$\cot a \cot b \pm 1 = \frac{\cos (a \mp b) \cot b}{\sin a \cos b} = \frac{\cos (a \mp b)}{\cos a \cos b}$$

§ 22. By separating the signs in the formula G, No. 1, and multiplying the separate parts, we obtain a formula for the difference of the squares of the tangents, that is very simple, and analogous in its nature to the original formula; we have

5 $$(\tan a + \tan b)(\tan a - \tan b) = \tan^2 a - \tan^2 b = \frac{\sin (a+b) \sin (a-b)}{\cos^2 a \cos^2 b}$$

And similar formulæ are deduced, with equal ease, from the other formulæ of the same character; they do not however appear to require, that their investigation be given here, in detail, and they are, besides, easily found in case they are needed.

CHAPTER V.

Combination of the Formulæ of the Sum, and difference of two Angles.

§ 23. Separating the signs in the formulæ F, No. 1 and 2, and combining them, by addition and subtraction, we obtain a series of simple formulæ, that are very useful in their practical application to calculation, viz.

$$\sin(a+b)+\sin(a-b)= \qquad \text{K}$$
$$\sin a\cos b+\cos a\sin b+\sin a\cos b-\sin b\cos a=2\sin a\cos b \qquad 1$$

$$\sin(a+b)-\sin(a-b)= \qquad 2$$
$$\sin a\cos b+\cos a\sin b-\sin a\cos b+\sin b\cos a=2\cos a\sin b$$

$$\cos(a-b)+\cos(a+b)= \qquad 3$$
$$\cos a\cos b+\sin a\sin b+\cos a\cos b-\sin a\sin b=2\cos a\cos b$$

$$\cos(a-b)-\cos(a+b)= \qquad 4$$
$$\cos a\cos b+\sin a\sin b-\cos a\cos b+\sin a\sin b=2\sin a\sin b$$

$$\sin(a\pm b)\pm\cos(a\pm b)=\sin a(\cos b\mp\sin b)\pm\cos a(\cos b\pm\sin b) \qquad 5$$

As this last formula does not present any peculiar interest, it is not deduced in detail; it may be found by a simple calculation.

§ 24. The addition of the two values of F, No. 3, with their signs changed, gives the following formulæ, by means of a very simple process of reduction:

$$\tan(a\pm b)+\tan(a\mp b)=\frac{\sin(a\pm b)}{\cos(a\pm b)}+\frac{\sin(a\mp b)}{\cos(a\mp b)}$$

Reducing to a common denominator

$$=\frac{\sin(a\pm b)\cos(a\mp b)+\sin(a\mp b)\cos(a\pm b)}{\cos(a\pm b)\cos(a\mp b)}$$

The numerator being $=\sin\big((a\pm b)+(a\mp b)\big)=\sin 2a$, and performing the multiplication in the denominator, we have the above.

$$=\frac{\sin 2a}{\cos^2 a\cos^2 b-\sin^2 a\sin^2 b}$$

And because, $\cos^2 b=1-\sin^2 b$; and, $\sin^2 a=1-\cos^2 a$; and by compensation,

$$\tan(a\pm b)+\tan(a\mp b)=\frac{\sin 2a}{\cos^2 a-\sin^2 b} \qquad 6$$

The subtraction of these same two expressions, gives a result exactly similar, with this exception: that the two terms of the numerator are separated by the sign —, instead of

$+$. It results from this: that, instead of the sine of the sum of the two angles $(a \pm b)$ and $(a \mp b)$ the numerator represents the sine of the difference of these angles; we then have as numerator,

$$\sin\big((a \pm b) - (a \mp b)\big) = \sin(\pm 2b) = \pm \sin 2b$$

As the denominator does not undergo any change, the definitive formula, which requires the same steps for its reduction as the preceding, becomes

7 $$\tan(a \pm b) - \tan(a \mp b) = \frac{\pm \sin 2b}{\cos^2 a - \sin^2 b}$$

If we now treat in the same manner the formula for the cotangents, F, No. 8, and pay attention to the fact, that the cotangents of small angles are greater than those of large angles; and therefore, as has been already remarked, the subtraction must be inverted. We have

$$\cot(a \mp b) + \cot(a \pm b) = \frac{\cos(a \mp b)}{\sin(a \mp b)} + \frac{\cos(a \pm b)}{\sin(a \pm b)}$$

$$= \frac{\cos(a \mp b)\sin(a \pm b) + \cos(a \pm b)\sin(a \mp b)}{\sin(a \mp b)\sin(a \pm b)}$$

The numerator is evidently the same as in formula 6, and the denominator is reduced to the difference of the squares of the two terms of the formula which gives the sine of the sum or difference of two angles; we then have, again, for the angle of the numerator,

$$\sin\big((a \pm b) + (a \mp b)\big) = \sin 2a$$

And the formula will, by applying reductions to the denominator, as before, ultimately become,

8 $$\cot(a \mp b) + \cot(a \pm b) = \frac{\sin 2a}{\cos^2 b - \cos^2 a}$$

Subtracting the same two formulæ, we obtain, as in the case of the tangent, a numerator that represents the difference of the angles, and consequently, has exactly the same value as in formula 7, except that the signs are inverted, in consequence of the inverted subtraction, that is to say, $(a \mp b) - (a \pm b) = \mp 2b$; and as the denominator remains the same as in formula 7, the final formula will become **K**

$$\cot(a \mp b) - \cot(a \pm b) = \frac{\mp \sin 2b}{\cos^2 b - \cos^2 a} \qquad 9$$

By a process precisely similar to that given above, and whose detail is omitted here, for the express purpose of giving the student an opportunity of exercise in operations of the sort, we may obtain the two following results:

$$\tan(a \pm b) + \cot(a \mp b) = \frac{\cos 2b}{\cos a \sin a \mp \sin b \cos b} \qquad 10$$

$$\tan(a \pm b) - (\cot a \mp b) = \frac{-\cos 2a}{\cos a \sin a \mp \sin b \cdot \cos b} \qquad 11$$

It is obvious, that more combinations of this sort may be made, from the corresponding formulæ.

§ 25. It will easily be seen, by inspecting the formulæ of § 23 and 24, that by dividing any one of them by any other of the corresponding formulæ, taking in § 24 those which have either the same numerator or the same denominator, we can obtain formulæ of the greatest simplicity on the one side, corresponding to expressions on the other side of the equation, that are apparently complicated. But it would be useless to make these combinations here, as they are of the greatest facility.

§ 26. The formulæ of the series F, give, by multiplication, the following results. The signs being separated, as has been done in the greater part of the formulæ of the preceding series K.

Multiplying F, No. 1, and separating the signs.

$$\begin{aligned} &\sin(a+b)\sin(a-b) \\ &\quad = (\sin a \cos b + \cos a \sin b)(\sin a \cos b - \sin b \cos a) \\ &\quad = \sin^2 a \cos^2 b - \cos^2 a \sin^2 b \end{aligned}$$

L. And substituting, according to series C, No. 4 and 5.

1 $\sin(a+b)\sin(a-b) = \sin^2 a - \sin^2 b$

2 $ = \cos^2 b - \cos^2 a$

Multiplying F, No. 2, with separation of the signs, and an analogous process.

$$
\begin{aligned}
\cos(a+b)\cos(a-b) &\\
&= (\cos a\cos b - \sin a\sin b)(\cos a\cos b + \sin a\sin b)\\
&= \cos^2 a\cos^2 b - \sin^2 a\sin^2 b
\end{aligned}
$$

3 $ = \cos^2 a - \sin^2 b$

4 $ = \cos^2 b - \sin^2 a$

By multiplying together, F, No. 3, separating the signs, and observing: that in conformity with B, No. 3, there is a division that always corresponds with a multiplication, because $\text{tang} = \dfrac{1}{\cot}$, we obtain the following results:

$$\tan(a+b)\tan(a-b) = \frac{\tan(a+b)}{\cot(a-b)} = \frac{\sin(a+b)\sin(a-b)}{\cos(a+b)\cos(a-b)}$$

Expressing the factors of the numerator and the denominator, multiplying them actually, and reducing, according to series C, No. 4 and 5, this formula is reduced to

5 $$\tan(a+b)\tan(a-b) = \frac{\tan(a+b)}{\cot(a-b)} = \frac{\sin^2 a - \sin^2 b}{\cos^2 b - \sin^2 a} = \frac{\cos^2 b - \cos^2 a}{\cos^2 a - \sin^2 b}$$

We obtain, in the same manner, the two following formulæ, which are, besides, already evident from the four first formulæ of the present series.

6 $$\cot(a+b)\cot(a-b) = \frac{\cot(a+b)}{\tan(a-b)} = \frac{\cos^2 b - \sin^2 a}{\sin^2 a - \sin^2 b} = \frac{\cos^2 a - \sin^2 b}{\cos^2 b - \sin^2 a}$$

$$\tan(a+b)\cot(a-b) = \frac{\tan(a+b)}{\tan(a-b)} = \frac{\sin a \cos a + \sin b \cos b}{\sin a \cos a - \sin b \cos b} \quad \text{L } 7$$

No. 5 and 6 evidently admit the variations in their numerator and denominator that are pointed out by the equality of the two preceding ones, No. 1 and 2, 3 and 4; and which are, besides, evident consequences of series C.

§ 27. After this explanation, the manner in which analogous formulæ for the secant and cosecant may be deduced, will be readily perceived; this introduction may, therefore, be considered as sufficient, particularly as we do not conceive it necessary to give every possible formulæ, but merely to point out an easy and systematic mode of obtaining them.

§ 28. By treating in the same manner those formulæ of the series F, which express the tangents, cotangents, secants and cosecants of the sum or difference of two angles, in terms of the tangent and cotangent of the simple angles, we might obtain a series of symetric formulæ in terms of the tangent and cotangent of the same simple angles. A great number of these are simple, and may be useful; but for the reason already stated, it will be sufficient merely to point out the method.

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CHAPTER VI.

Trigonometric Functions, that express the Functions of Simple Angles, in terms of the Functions of Compound Angles.

§ 29. As it has always been assumed, in the preceding formulæ, that $a \geq b$, it being natural to make such an assumption in announcing any two quantities whose value and ratio is indeterminate, it follows from the principles of algebra, that

$$a = \tfrac{1}{2}(a+b) + \tfrac{1}{2}(a-b); \; b = \tfrac{1}{2}(a+b) - \tfrac{1}{2}(a-b)$$

Applying these denominations to the formulæ of the series F, and limiting the investigation to the sine, cosine, and tangent, (which is sufficient to exhibit the principles of this operation, and to lead to formulæ of general application, in a short and easy manner,) we obtain in succession the following trigonometric functions, viz.

M By F No. 1 will be obtained,

$$\sin a = \sin\left(\tfrac{1}{2}(a+b) + \tfrac{1}{2}(a-b)\right)$$

1 $$= \sin\tfrac{1}{2}(a+b)\cos\tfrac{1}{2}(a-b) + \cos\tfrac{1}{2}(a+b)\sin\tfrac{1}{2}(a-b)$$

and also,

2 $$\sin b = \sin\tfrac{1}{2}(a+b)\cos\tfrac{1}{2}(a-b) - \cos\tfrac{1}{2}(a+b)\sin\tfrac{1}{2}(a-b)$$

By F No. 2 will be obtained,

3 $$\cos a = \cos\tfrac{1}{2}(a+b)(\cos\tfrac{1}{2}(a-b) - \sin\tfrac{1}{2}(a+b)\sin\tfrac{1}{2}(a-b)$$

and

4 $$\cos b = \cos\tfrac{1}{2}(a+b)\cos\tfrac{1}{2}(a-b) + \sin\tfrac{1}{2}(a+b)\sin\tfrac{1}{2}(a-b)$$

By F No. 4 will be obtained $\tan a = \tan\left(\tfrac{1}{2}(a+b) + \tfrac{1}{2}(a-b)\right)$

5 $$= \frac{1 + \tan\tfrac{1}{2}(a-b)\cot\tfrac{1}{2}(a+b)}{\cot\tfrac{1}{2}(a+b) - \tan\tfrac{1}{2}(a-b)}$$

6 and likewise $$\tan b = \frac{1 - \tan\tfrac{1}{2}(a-b)\cot\tfrac{1}{2}(a+b)}{\cot\tfrac{1}{2}(a+b) + \tan\tfrac{1}{2}(a-b)}$$

It will be at once seen, that the formulæ 5, 6, 7, of the same series, might be also employed for this purpose, and would lead to analogous results. The above formulæ, 5 and 6, naturally give the cotangent by a simple inversion.

§ 30. If we now combine these formulæ in the same manner, and by the same rules as the preceding, we shall obtain a series of formulæ that are much more simple, and are susceptible of becoming general for every proportional value of the angles.

By addition, and the simple compensation of the signs of the second terms of the sines and cosines, we obtain, By adding No. 1 and 2, or M

$$\sin a + \sin b = 2 \sin \tfrac{1}{2}(a + b) \cos \tfrac{1}{2}(a - b) \qquad 7$$

By adding No. 3 and 4, or

$$\cos a + \cos b = 2 \cos \tfrac{1}{2}(a + b) \cos \tfrac{1}{2}(a - b)$$

By adding No. 5 and 6, or $\tan a + \tan b =$ 8

$$\frac{1 + \tan \frac{1}{2}(a - b) \cot \frac{1}{2}(a + b)}{\cot \frac{1}{2}(a + b) - \tan \frac{1}{2}(a - b)} + \frac{1 - \tan \frac{1}{2}(a - b) \cot \frac{1}{2}(a + b)}{\cot \frac{1}{2}(a + b) + \tan \frac{1}{2}(a - b)}$$

And by reducing this to a common denominator and compensating:

$$\tan a + \tan b = \frac{2 \tan \frac{1}{2}(a + b)\left(1 + \tan^2 \frac{1}{2}(a - b)\right)}{1 - \tan^2 \frac{1}{2}(a - b) \tan^2 \frac{1}{2}(a + b)} \qquad 9$$

By subtraction, and a process exactly analogous to the above, we obtain;

By subtracting No. 2 from No. 1, or

$$\sin a - \sin b = 2 \sin \tfrac{1}{2}(a - b) \cos \tfrac{1}{2}(a + b) \qquad 10$$

By subtracting No. 3 from No. 4, or

$$\cos b - \cos a = 2 \sin \tfrac{1}{2}(a + b) \sin \tfrac{1}{2}(a - b) \qquad 11$$

By subtracting No. 6 from No. 5, or

$$\tan a - \tan b = \frac{2 \cot \frac{1}{2}(a - b)\left(1 + \cot^2 \frac{1}{2}(a + b)\right)}{\cot^2 \frac{1}{2}(a + b) \cot^2 \frac{1}{2}(a - b) - 1} \qquad 12$$

We further obtain, by multiplication, as follows:

Multiplying No. 1 and 2, or, $\sin a \sin b$

$$= \sin^2 \tfrac{1}{2}(a + b) \cos^2 \tfrac{1}{2}(a - b) - \cos^2 \tfrac{1}{2}(a + b) \sin^2 \tfrac{1}{2}(a - b)$$

$$= \sin^2 \tfrac{1}{2}(a + b) - \sin^2 \tfrac{1}{2}(a - b) \qquad 13$$

$$= \cos^2 \tfrac{1}{2}(a - b) - \cos^2 \tfrac{1}{2}(a + b) \qquad 14$$

Multiplying No. 3 and 4, or, $\cos a \cos b$

$$= \cos^2 \tfrac{1}{2}(a + b) \cos^2 \tfrac{1}{2}(a - b) - \sin^2 \tfrac{1}{2}(a + b) \sin^2 \tfrac{1}{2}(a - b)$$

$$= \cos^2 \tfrac{1}{2}(a + b) - \sin^2 \tfrac{1}{2}(a - b) \qquad 15$$

$$= \cos^2 \tfrac{1}{2}(a - b) - \sin^2 \tfrac{1}{2}(a + b) \qquad 16$$

Multiplying No. 5 and 6, or

$$\tan a \tan b = \frac{1 - \tan^2 \frac{1}{2}(a - b) \cot^2 \frac{1}{2}(a + b)}{\cot^2 \frac{1}{2}(a + b) - \tan^2 \frac{1}{2}(a - b)} \tag{17}$$

For since the terms of the numerators and denominators in the two fractions are the same; being in the one a sum, in the other a difference, their product is the difference of their squares.

By the division of the similar functions of the two angles we obtain, as follows:

Dividing No. 1 by No. 2, or

$$\frac{\sin a}{\sin b} = \frac{\sin \frac{1}{2}(a + b) \cos \frac{1}{2}(a - b) + \cos \frac{1}{2}(a + b) \sin \frac{1}{2}(a - b)}{\sin \frac{1}{2}(a + b) \cos \frac{1}{2}(a - b) - \sin \frac{1}{2}(a - b) \cos \frac{1}{2}(a + b)}$$

And dividing all the terms by the first term,

$$\frac{\sin a}{\sin b} = \frac{1 + \cot \frac{1}{2}(a + b) \tan \frac{1}{2}(a - b)}{1 - \cot \frac{1}{2}(a + b) \tan \frac{1}{2}(a - b)} \tag{18}$$

It is evident that we also have:

$$\frac{\sin a}{\sin b} = \frac{\tan \frac{1}{2}(a + b) \cot \frac{1}{2}(a - b) + 1}{\tan \frac{1}{2}(a + b) \cot \frac{1}{2}(a - b) - 1} \tag{19}$$

Dividing No. 3 by No. 4,

$$\frac{\cos a}{\cos b} = \frac{\cos \frac{1}{2}(a + b) \cos \frac{1}{2}(a - b) - \sin \frac{1}{2}(a + b) \sin \frac{1}{2}(a - b)}{\cos \frac{1}{2}(a + b) \cos \frac{1}{2}(a - b) + \sin \frac{1}{2}(a + b) \sin \frac{1}{2}(a - b)}$$

And dividing by the two terms successively, as before:

$$\frac{\cos a}{\cos b} = \frac{1 - \tan \frac{1}{2}(a + b) \tan \frac{1}{2}(a - b)}{1 + \tan \frac{1}{2}(a + b) \tan \frac{1}{2}(a - b)} \tag{20}$$

$$= \frac{\cot \frac{1}{2}(a + b) \cot \frac{1}{2}(a - b) - 1}{\cot \frac{1}{2}(a + b) \cot \frac{1}{2}(a - b) + 1} \tag{21}$$

The division of the *sine* and *cosine* of the same angle, evidently gives the formulæ 5 and 6, and cross divisions give analogous formulæ, expressed in terms of *tangent* and *cotangent,* which may be readily found when needed.

Dividing the *tangents*, we have by the formulæ 5 and 6,

$$\frac{\tan a}{\tan b} = \frac{\left(1+\tan \tfrac{1}{2}(a-b)\cot \tfrac{1}{2}(a+b)\right)\left(\cot \tfrac{1}{2}(a+b)+\tan \tfrac{1}{2}(a-b)\right)}{\left(1-\tan \tfrac{1}{2}(a-b)\cot \tfrac{1}{2}(a+b)\right)\left(\cot \tfrac{1}{2}(a+b)-\tan \tfrac{1}{2}(a-b)\right)}$$

$$= \frac{\cot \tfrac{1}{2}(a+b)+\tan \tfrac{1}{2}(a-b)+\tan \tfrac{1}{2}(a-b)\cot^2 \tfrac{1}{2}(a+b)+\cot \tfrac{1}{2}(a+b)\tan^2 \tfrac{1}{2}(a-b)}{\cot \tfrac{1}{2}(a+b)-\tan \tfrac{1}{2}(a-b)-\tan \tfrac{1}{2}(a-b)\cot^2 \tfrac{1}{2}(a+b)+\cot \tfrac{1}{2}(a+b)\tan^2 \tfrac{1}{2}(a-b)}$$

$$= \frac{\cot \tfrac{1}{2}(a+b)\left(1+\tan^2 \tfrac{1}{2}(a-b)\right)+\tan \tfrac{1}{2}(a-b)\left(1+\cot^2 \tfrac{1}{2}(a+b)\right)}{\cot \tfrac{1}{2}(a+b)\left(1+\tan^2 \tfrac{1}{2}(a-b)\right)-\tan \tfrac{1}{2}(a-b)\left(1+\cot^2 \tfrac{1}{2}(a+b)\right)}$$

And from series C, No. 6 and 7,

$$= \frac{\cot \tfrac{1}{2}(a+b)\sec^2 \tfrac{1}{2}(a-b)+\tan \tfrac{1}{2}(a-b)\operatorname{cosec}^2 \tfrac{1}{2}(a+b)}{\cot \tfrac{1}{2}(a+b)\sec^2 \tfrac{1}{2}(a-b)-\tan \tfrac{1}{2}(a-b)\operatorname{cosec}^2 \tfrac{1}{2}(a+b)}$$

Taking from series B, No. 1 and 2, squared; that is to say, $\sec^2 = \dfrac{1}{\cos^2}$; $\operatorname{cosec}^2 = \dfrac{1}{\sin^2}$; which values being introduced in the formula, and the terms reduced to a common denominator, that is compensated in the numerator and denominator, the formula is reduced to the following, viz:

$$= \frac{\cot \tfrac{1}{2}(a+b)\sin^2 \tfrac{1}{2}(a+b)+\tan \tfrac{1}{2}(a-b)\cos^2 \tfrac{1}{2}(a-b)}{\cot \tfrac{1}{2}(a+b)\sin^2 \tfrac{1}{2}(a+b)-\tan \tfrac{1}{2}(a-b)\cos^2 \tfrac{1}{2}(a-b)}$$

Which by compensation is finally reduced to

$$\frac{\tan a}{\tan b} = \frac{\sin \tfrac{1}{2}(a+b)\cos \tfrac{1}{2}(a+b)+\sin \tfrac{1}{2}(a-b)\cos \tfrac{1}{2}(a-b)}{\sin \tfrac{1}{2}(a+b)\cos \tfrac{1}{2}(a+b)-\sin \tfrac{1}{2}(a-b)\cos \tfrac{1}{2}(a-b)} \qquad 22$$

a formula that is rather curious than useful, and analogous to L, No. 7.

§ 31 The preceding formulæ from No. 7 may be combined for different uses; but as we may now assume the student to possess a sufficient knowledge of this method of deducing compound trigonometric functions, we shall only mention a few obtained by division, that are of such frequent use, that it would be improper to omit them.

N Dividing No. 7 by No. 8, or

1 $$\frac{\sin a + \sin b}{\cos a + \cos b} = \frac{2 \sin \frac{1}{2}(a+b) \cos \frac{1}{2}(a-b)}{2 \cos \frac{1}{2}(a+b) \cos \frac{1}{2}(a-b)} = \tan \tfrac{1}{2}(a+b)$$

Dividing No. 10 by No. 11, or

2 $$\frac{\sin a - \sin b}{\cos b - \cos a} = \frac{2 \sin \frac{1}{2}(a-b) \cos \frac{1}{2}(a+b)}{2 \sin \frac{1}{2}(a+b) \sin \frac{1}{2}(a-b)} = \cot \tfrac{1}{2}(a+b)$$

Dividing No. 7 by No. 10, or

3 $$\frac{\sin a + \sin b}{\sin a - \sin b} = \frac{2 \sin \frac{1}{2}(a+b) \cos \frac{1}{2}(a-b)}{2 \sin \frac{1}{2}(a-b) \cos \frac{1}{2}(a+b)} = \frac{\tan \frac{1}{2}(a+b)}{\tan \frac{1}{2}(a-b)}$$

Dividing No. 8 by No. 11, or

4 $$\frac{\cos a + \cos b}{\cos b - \cos a} = \frac{2 \cos \frac{1}{2}(a+b) \cos \frac{1}{2}(a-b)}{2 \sin \frac{1}{2}(a+b) \sin \frac{1}{2}(a-b)} = \frac{\cot \frac{1}{2}(a+b)}{\tan \frac{1}{2}(a-b)}$$

Dividing No. 7 by No. 11, or

5 $$\frac{\sin a + \sin b}{\cos b - \cos a} = \frac{2 \sin \frac{1}{2}(a+b) \cos \frac{1}{2}(a-b)}{2 \sin \frac{1}{2}(a+b) \sin \frac{1}{2}(a-b)} = \cot \tfrac{1}{2}(a-b)$$

Dividing No. 10 by No. 8, or

6 $$\frac{\sin a - \sin b}{\cos b + \cos a} = \frac{2 \sin \frac{1}{2}(a-b) \cos \frac{1}{2}(a+b)}{2 \cos \frac{1}{2}(a-b) \cos \frac{1}{2}(a+b)} = \tan \tfrac{1}{2}(a-b)$$

CHAPTER VII.

General Formulæ for the Multiples of Angles.

§ 32 The formulæ of the preceding chapter, series **M**, are susceptible of the utmost generalization, and give, in this way, general values of the trigonometric functions of the multiples of angles, expressed in terms of the functions of the simple angle or some inferior multiple. To do this we need only assign to a, and b, a certain relative value, expressed in general terms; we may then transcribe the formulæ in the form they assume under this supposition, transpose and reduce them by known compensations.

Let us assume for the two angles, a, and b, the following proportional multiples: of a, make $n\ a$; and of b, make $(n-2)\ a$.

Taking the formulæ 7, 8, 9, 10, 11, 12; transposing in the first place, in order to abridge the operation, all the second terms of each equation; we have as follows:

From M, 0

No. 7; $\sin na = 2\sin(n-1)a\cos a - \sin(n-2)a$ 1

8; $\cos na = 2\cos(n-1)a\cos a - \cos(n-2)a$ 2

$$9;\ \tan na = \frac{2\tan(n-1)a\,(1+\tan^2 a)}{1-\tan^2(n-1)a\tan^2 a} - \tan(n-2)a =$$

$$\frac{2\tan(n-1)a\sec^2 a - \tan(n-2)a\left(1-\tan^2(n-1)a\tan^2 a\right)}{1-\tan^2(n-1\,a\tan^2 a} \tag{3}$$

From M,

No. 10; $\sin na = 2\cos(n-1)a\sin a + \sin(n-2)a$ 4

11; $\cos na = -2\sin(n-1)a\sin a + \cos(n-2)a$ 5

$$12;\ \tan na = \frac{2\cot a\left(1+\cot^2(n-1)a\right)}{\cot^2(n-1)a\cot^2 a - 1} - \frac{1}{\cot(n-2)a} =$$

$$\frac{2\cot a\operatorname{cosec}^2(n-1)a\cot(n-2)a - \cot^2(n-1)a\cot^2 a + 1}{\cot(n-2)a\left(\cot^2(n-1)a - \cot^2 a - 1\right)} \tag{6}$$

Notwithstanding the complicated appearance of No. 3 and 6, they may be reduced to forms comparatively simple in their application to numbers substituted for n.

We moreover have, by the formulæ 5 and 6 of series M, expressions that fulfil (though in part only) the same object.

O By transforming

7 **M** No. 5; $\tan na = \dfrac{1 + \tan a \tan (n - 1) a}{\cot (n - 1) a - \tan a}$

8 6; $\tan (n - 2) a = \dfrac{1 - \tan a \cot (n - 1) a}{\cot (n - 1) a + \tan a}$

§ 33 If we give to n, the value of the several terms of the series of natural numbers in succession, we may obtain from the preceding formulæ two series of expressions for multiple angles in a regular ascending order. Thus we have, from the formulæ O, No. 1 and 4, by successive assumptions of the

P value of n, = 1, 2, 3, &c.

$\sin a = \sin a \qquad = \sin a$

$\sin 2a = 2 \sin a \cos a \qquad = 2 \cos a \sin a$

1 $\sin 3a = 2 \sin 2a \cos a - \sin a \qquad = 2 \cos 2a \sin a + \sin a$

$\sin 4a = 2 \sin 3a \cos a - \sin 2a \qquad = 2 \cos 3a \sin a + \sin 2a$

2 $\sin 5a = 2 \sin 4a \cos a - \sin 3a \qquad = 2 \cos 4a \sin a + \sin 3a$

$\sin 6a = 2 \sin 5a \cos a - \sin 4a \qquad = 2 \sin 5a \sin a + \sin 4a$

$\sin 7a =$ &c.

From the formulæ 2 and 5, we obtain the following series for the cosines:

$\cos a = \cos a \qquad = \cos a$

3 $\cos 2a = \cos^2 a - 1 \qquad = - 2 \sin^2 a + 1$

$\cos 3a = 2 \cos 2a \cos a - \cos a \qquad = - 2 \sin 2a \sin a + \cos a$

$\cos 4a = 2 \cos 3a \cos a - \cos 2a \qquad = - 2 \sin 3a \sin a + \cos 2a$

$\cos 5a = 2 \cos 4a \cos a - \cos 3a \qquad = - 2 \sin 4a \sin a + \cos 3a$

4 $\cos 6a = 2 \cos 5a \cos a - \cos 4a \qquad = - 2 \sin 5a \sin a + \cos 4a$

$\cos 7a =$ &c.

We obtain for the tangents from the formula O, No. 3,

$$\left.\begin{array}{l}
\tan a = \tan a \\[2ex]
\tan 2a = \dfrac{2 \tan a}{1 - \tan^2 a} \\[3ex]
\tan 3a = \dfrac{2 \tan 2a + 2 \tan 2a \tan^2 a - \tan a + \tan^2 2a \tan^2 a}{1 - \tan^2 2a \tan^2 a} \\[3ex]
\tan 4a = \dfrac{2 \tan 3a + 2 \tan 3a \tan^2 a - \tan 2a + \tan^2 3a \tan 2a \tan^2 a}{1 - \tan^2 3a \tan^2 a} \\[3ex]
\tan 5a = \dfrac{2 \tan 4a + 2 \tan 4a \tan^2 a - \tan 3a + \tan^2 4a \tan 3a \tan^2 a}{1 - \tan^2 4a \tan^2 a} \\[3ex]
\tan 6a = \dfrac{2 \tan 5a + 2 \tan 5a \tan^2 a - \tan 4a + \tan^2 5a \tan 4a \tan^2 a}{1 - \tan^2 5a \tan^2 a} \\[3ex]
\tan 7a = \&c.
\end{array}\right\} 5$$

From the formula O, No. 6, we have

$$\left.\begin{array}{l}
\tan a = \dfrac{1}{\cot a} \\[3ex]
\tan 2a = \dfrac{2 \cot a}{\cot^2 a - 1} \\[3ex]
\tan 3a = \dfrac{2 \cot^2 a + \cot^3 a \cot^2 2a + 1}{\cot^3 a \cot^2 2a - \cot a} \\[3ex]
\tan 4a = \dfrac{2 \cot a \cot 2a + 2 \cot a \cot 2a \cot^2 3a - \cot^2 a \cot^2 3a + 1}{\cot 2a \cot^2 3a \cot^2 a - \cot 2a} \\[3ex]
\tan 5a = \dfrac{2 \cot a \cot 3a + 2 \cot a \cot 3a \cot^3 4a - \cot^2 a \cot^2 4a + 1}{\cot 3a \cot^2 4a \cot^2 a - \cot 3a} \\[3ex]
\tan 6a = \dfrac{2 \cot a \cot 4a + 2 \cot a \cot 4a \cot^2 5a - \cot^2 a \cot^2 5a + 1}{\cot 4a \cot^2 5a \cot^3 a - \cot 4a} \\[3ex]
\tan 7a = \&c.
\end{array}\right\} 6$$

All these formulæ may be changed into such as have no other functions involved, than those of the simple angle; by substituting successively in the formulæ of the multiple angles, the values found for their different factors, in the expressions that precede them. There will result, as may be foreseen, a double series of formulæ, which will follow regular laws. From these may be deduced, finally, a general law for the combination of the trigonometric functions of multiple angles, in terms of the simple angle. It may readily be conceived, that, in consequence of the multiplicity of formulæ furnished by the trigonometric functions, many similar series may be made, varied in a high degree, and adapted to every varying purpose. It will be also readily seen, that such series must finally lead to a general law, in the same way that the binomial theorem furnishes the law that governs the combination of the different powers of two quantities.

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CHAPTER VIII.

Formulæ for Double Angles, expressed in terms of the Functions of the Simple Angles, and for Half Angles, expressed in terms of the Functions of the Whole Angle.

§ 34. Among the formulæ that express the functions of multiple angles, in terms of the functions of the simple angle; those which give the functions of the double angle, in terms of the functions of the simple angle; and the functions of the half angle, in terms of the whole angle; are of such frequent use, that it is proper to treat of them separately, and to collect the results for future use.

In order to obtain them, it is sufficient to assume, in the formulæ of the series F, with the sign +, the value of $a = b$, whence $a + b = 2a$; to transcribe them here with the reductions produced by the calculation itself, and the formulæ of the series B and C. In order to abridge the work, and present at one view these formulæ in a short table, by which the frequent use that will be made of them may be facilitated, we shall suppose that recourse is had to these series for the explanation of the requisite operations, and that it is not necessary to refer to them in every particular instance.

Q

By F No. 1, is obtained; $\sin 2a = 2 \sin a \cos a$ 1

2, $\cos 2a = \cos^2 a - \sin^2 a$ 2

$= 1 - 2 \sin^2 a$ 3

$= 2 \cos^2 a - 1$ 4

4, $\tan 2a = \dfrac{2}{\cot a - \tan a}$ 5

6, $= \dfrac{2 \tan a}{1 - \tan^2 a}$ 6

7, $= \dfrac{2 \cot a}{\cot^2 a - 1}$ 7

8, $\cot 2a = \dfrac{\cot a - \tan a}{2}$ 8

11, $= \dfrac{1 - \tan^2 a}{2 \tan a}$ 9

12, $= \dfrac{\cot^2 a - 1}{2 \cot a}$ 10

The three last formulæ being evidently a mere inversion of the three foregoing, as might be expected.

By F, No. 13, is obtained; $\sec 2a = \dfrac{1}{\cos^2 a - \sin^2 a}$ 11

and hence, $= \dfrac{1}{2 \cos^2 a - 1}$ 12

$= \dfrac{1}{1 - 2 \sin^2 a}$ 13

Q

14 By F, No. 16, $\sec 2a = \frac{\sec^2 a}{1 - \tan^2 a}$;

15 17, $= \frac{\operatorname{cosec}^2 a}{\cot^2 a - 1}$

16 20, $= \frac{1 + \tan^2 a}{1 - \tan^2 a}$

17 21, $= \frac{1 + \cot^2 a}{\cot^2 a - 1}$

18 22, $\operatorname{cosec} 2a = \frac{1}{2 \sin a \cos a}$

19 25, $= \frac{\operatorname{cosec}^2 a}{2 \cot a}$

20 26, $= \frac{\sec^2 a}{2 \tan a}$

21 29, $= \frac{1 + \cot^2 a}{2 \cot a}$

22 30, $= \frac{1 + \tan^2 a}{2 \tan a}$

The formulæ for the secant give new formulæ for the cosine, by changing the numerator into denominator, and inversely. The formulæ for the cosecant give new formulæ for the sine, by the same process.

The formulæ F, No. 5, and 6; 18, and 19; 27, and 28, have not been employed, because they furnish less simple formulæ, and it has not been considered necessary to swell this table by using them.

§ 35. For the expression of the sine of half the angle, in terms of the functions of the whole angle, we first obtain, by the transposition of the formula Q, No. 3, writing $\frac{1}{2}a$, instead of a; and a, instead of $2a$.

$$2 \sin^2 \tfrac{1}{2} a = 1 - \cos a \qquad \text{whence} \qquad \text{R}$$

$$\sin \tfrac{1}{2} a = \frac{(1 - \cos a)^{\frac{1}{2}}}{\sqrt{2}} \qquad 1$$

Substituting, in the first of these formulæ, the values taken from C, No. 5, we have,

$$2 \sin^2 \tfrac{1}{2} a = 1 - (1 - \sin^2 a)^{\frac{1}{2}}$$

The part under the radicals, may be considered as the difference of two squares, and expressed by the product of the sum and difference of its roots. This transforms it into

$$2 \sin^2 \tfrac{1}{2} a = 1 - (1 + \sin a)^{\frac{1}{2}} (1 - \sin a)^{\frac{1}{2}}$$

Adding, on the right hand side of the equation,

$$\frac{\sin a}{2} - \frac{\sin a}{2} = o,$$ which does not change its value, and making $1 = \tfrac{1}{2} + \tfrac{1}{2}$, it becomes

$$2 \sin^2 \tfrac{1}{2} a = \tfrac{1}{2} + \frac{\sin a}{2} + \tfrac{1}{2} - \frac{\sin a}{2.} - (1 + \sin a)^{\frac{1}{2}} (1 - \sin a)^{\frac{1}{2}}$$

$$= \frac{1 - \sin a}{2} + \frac{1 - \sin a}{2} - (1 + \sin a)^{\frac{1}{2}} (1 - \sin a)^{\frac{1}{2}}$$

As this forms a complete square, we may extract the root, which gives us

$$\sin \tfrac{1}{2} a \sqrt{2} = \frac{(1 + \sin a)^{\frac{1}{2}}}{\sqrt{2}} - \frac{(1 - \sin a)^{\frac{1}{2}}}{\sqrt{2}}$$

and dividing by $\sqrt{2}$,

$$\sin \tfrac{1}{2} a = \tfrac{1}{2} (1 + \sin a)^{\frac{1}{2}} - \tfrac{1}{2} (1 - \sin a)^{\frac{1}{2}} \qquad 2$$

We obtain for the expression of the cosine, by taking the formula F, No. 4, and treating it in exactly the same manner that we have done for the sine, the following results, in succession; viz.

$$2 \cos^2 \tfrac{1}{2} a = 1 + \cos a$$

R whence

3 $$\cos \tfrac{1}{2} a = \frac{(1 + \cos a)^{\frac{1}{2}}}{\sqrt{2}}$$

And for the formula analogous to No. 2, of this series,

$$\begin{aligned} 2 \cos^2 \tfrac{1}{2} a &= 1 + (1 - \sin^2 a)^{\frac{1}{2}} \\ &= 1 + (1 + \sin a)^{\frac{1}{2}} (1 - \sin a)^{\frac{1}{2}} \\ &= \frac{1 + \sin a}{2} + \frac{1 - \sin a}{2} + (1 + \sin a)^{\frac{1}{2}} (1 - \sin a)^{\frac{1}{2}} \end{aligned}$$

$$\cos \tfrac{1}{2} a \sqrt{2} = \frac{(1 + \sin a)^{\frac{1}{2}}}{\sqrt{2}} + \frac{(1 - \sin a)^{\frac{1}{2}}}{\sqrt{2}}$$

4 $$\cos \tfrac{1}{2} a = \tfrac{1}{2} (1 + \sin a)^{\frac{1}{2}} + \tfrac{1}{2} (1 - \sin a)^{\frac{1}{2}}$$

We obtain an expression for the tangent, in a very simple way, by dividing the expression for the sine by that for the cosine, thus:

5 $$\tan \tfrac{1}{2} a = \frac{\sin \frac{1}{2} a}{\cos \frac{1}{2} a} = \frac{(1 - \cos a)^{\frac{1}{2}}}{(1 + \cos a)^{\frac{1}{2}}}$$

Multiplying both numerator and denominator by $(1 - \cos a)^{\frac{1}{2}}$ we obtain

6 $$\tan \tfrac{1}{2} a = \frac{1 - \cos a}{(1 + \cos a)^{\frac{1}{2}} (1 - \cos a)^{\frac{1}{2}}} = \frac{1 - \cos a}{(1 - \cos^2 a)^{\frac{1}{2}}} = \frac{1 - \cos a}{\sin a}$$

Or, multiplying in the same way by $(1 + \cos a)^{\frac{1}{2}}$

7 $$\tan \tfrac{1}{2} a = \frac{(1 - \cos a)^{\frac{1}{2}} (1 + \cos a)^{\frac{1}{2}}}{1 + \cos a} = \frac{(1 - \cos^2 a)^{\frac{1}{2}}}{1 + \cos a} = \frac{\sin a}{1 + \cos a}$$

By the division of No. 2 of this series by No. 4, we obtain:

8 $$\tan \tfrac{1}{2} a = \frac{(1 + \sin a)^{\frac{1}{2}} - (1 - \sin a)^{\frac{1}{2}}}{(1 + \sin a)^{\frac{1}{2}} + (1 - \sin a)^{\frac{1}{2}}} = \frac{1 + \sin a - \cos a}{1 + \sin a + \cos a}$$

CHAPTER IX.

Trigonometric Functions of Compound Angles, of which one Part has a Determinate Value.

§ 36. In the formulæ of the series F, whence we have deduced the elementary formulæ for angles in a *determinate ratio* to each other, we may also assume: that one of the angles has a *determinate value;* and deduce useful formulæ.

In this case it will be proper to make use of angles of which the trigonometric functions that are to be employed are simple quantities. Taking then, as in series E, $h = 1$, and comparing the values of d and k, for the several values of the angles; keeping also in mind the principles that have been already explained, namely, that

$$\sin 45^\circ = \cos 45^\circ, \text{ and } \tan 45^\circ = \cot 45^\circ;\ \sec 45^\circ = \operatorname{cosec} 45^\circ;$$

it will be found, by the application of the formula,

$$h^2 = d^2 + k^2,$$

S

that

$$\sin 45^\circ = \cos 45^\circ = \frac{1}{\sqrt{2}} \tag{1}$$

$$\tan 45^\circ = \cot 45^\circ = 1 \tag{2}$$

$$\sec 45^\circ = \operatorname{cosec} 45^\circ = \sqrt{2} \tag{3}$$

It being a property of the circle, that the chord of the sixth part of the circumference is equal to radius; and that the perpendicular drawn from the centre upon this chord divides it into two equal parts, which represent the d, of the half of this sixth part of the circumference; that is to say, that as the sixth part of the circle is 60°, these parts represent each the value of d, for an angle of 30°; we have in addition to the above,

$$\sin 30^\circ = \cos 60^\circ = \tfrac{1}{2} \tag{4}$$

And because $\sin^2 a = 1 - \cos^2 a$

S

5 $$\cos 30^\circ = \sin 60^\circ = \left(1 - (\tfrac{1}{2})^2\right)^{\frac{1}{2}} = \frac{\sqrt{3}}{2}$$

6 $$\tan 30^\circ = \cot 60^\circ = \frac{1}{\sqrt{3}}$$

7 $$\cot 30^\circ = \tan 60^\circ = \sqrt{3}$$

8 $$\sec 30^\circ = \operatorname{cosec} 60^\circ = \frac{2}{\sqrt{3}}$$

9 $$\operatorname{cosec} 30^\circ = \sec 60^\circ = 2$$

There are a number of other similar values, to be found in the trigonometric functions, the research of which is called Rational Trigonometry. But we have no room to inquire into these, in this treatise.

§ 37. The formulæ that employ these determinate angles, are easily deduced from those of the series F.

Assuming, in the first place, that $a = 45^\circ$, we have

10 By F, No. 1; $$\sin(45^\circ \pm b) = \cos(45^\circ \mp b) = \frac{\cos b \pm \sin b}{\sqrt{2}}$$

11 2; $$\cos(45^\circ \pm b) = \sin(45^\circ \mp b) = \frac{\cos b \mp \sin b}{\sqrt{2}}$$

12 3; $$\tan(45^\circ \pm b) = \cot(45^\circ \mp b) = \frac{\cos b \pm \sin b}{\cos b \mp \sin b}$$

13 4; $$= \qquad = \frac{1 \pm \tan b}{1 \mp \tan b}$$

14 5; $$= \qquad = \frac{\cot b \pm 1}{\cot b \mp 1}$$

These formulæ also give those of the other trigonometric functions, by their inversion, which therefore are not repeated here.

Substituting these values in succession, in the formulæ of the series K, dividing by $\sqrt{2}$ in the two first, and reducing the two last, as indicated in the formulæ Q, No. 3 and 4, we have:

By K, No. 1 and 3;

$$\frac{\sin(45^\circ+b)+\sin(45^\circ-b)}{\sqrt{2}}=\cos b=\frac{\cos(45^\circ+b)+\cos(45^\circ-b)}{\sqrt{2}} \tag{15}$$

By K, No. 2 and 4;

$$\frac{\sin(45^\circ+b)-\sin(45^\circ-b)}{\sqrt{2}}=\sin b=\frac{\cos(45^\circ-b)-\cos(45^\circ+b)}{\sqrt{2}} \tag{16}$$

By K, No. 6 and 8;

$$\tan(45^\circ+b)+\tan(45^\circ-b)=\frac{2}{\cos 2b}=\cot(45^\circ-b)+\cot(45^\circ+b) \tag{17}$$

By K, No. 7 and 9;

$$\tan(45^\circ+b)-\tan(45^\circ-b)=\frac{2\sin 2b}{\cos 2b}=\cot(45^\circ-b)-\cot(45^\circ+b) \tag{18}$$

If we substitute, in these same formulæ, the values of the sines, cosines, tangents, and cotangents, of the angles of 60°, and 30°, ascribing these values to the angle a, we obtain the following formulæ:

By F, No. 1; $$\sin(30^\circ \pm b)=\frac{\cos b \pm \sin b\sqrt{3}}{2}=\sin(60^\circ \pm b) \tag{19}$$

2; $$\cos(30^\circ \pm b)=\frac{\cos b\sqrt{3} \mp \sin b}{2}=\sin(60^\circ \pm b) \tag{20}$$

3; $$\tan(30^\circ \pm b)=\frac{\cos b \pm \sin b\sqrt{3}}{\cos b\sqrt{3} \mp \sin b}=\cot(60^\circ \pm b) \tag{21}$$

4; $$=\frac{1 \pm \tan b\sqrt{3}}{\sqrt{3} \mp \tan b} \tag{22}$$

5; $$=\frac{\cot b \pm \sqrt{3}}{\cot b\sqrt{3} \mp 1} \tag{23}$$

24 By K, No. 1; $\cos b = \sin(30^\circ + b) + \sin(30^\circ - b)$

$= \cos(60^\circ + b) + \cos(60^\circ - b)$

25 2; $\sin b \sqrt{3} = \sin(30^\circ + b) - \sin(30^\circ - b)$

$= \cos(60^\circ - b) - \cos(60^\circ + b)$

26 3; $\cos b \sqrt{3} = \cos(30^\circ + b) + \cos(30^\circ - b)$

$= \sin(60^\circ + b) + \sin(60^\circ - b)$;

27 4; $\sin b = \cos(30^\circ - b) - \cos(30^\circ + b)$

$= \sin(60^\circ + b) - \sin(60^\circ - b)$

28 6; $\dfrac{2\sqrt{3}}{3 - 4\sin^2 b} = \tan(30^\circ \pm b) + \tan(30^\circ \mp b)$

29 7; $\dfrac{4\sin 2b}{3 - 4\sin^2 b} = \tan(30^\circ + b) - \tan(30^\circ - b)$

30 8; $\dfrac{2\sqrt{3}}{4\cos^2 b - 3} = \cot(30^\circ \mp b) + \cot(30^\circ \pm b)$

31 9; $\dfrac{4\sin 2b}{4\cos^2 b \quad 3} = \cot(30^\circ - b) - \cot(30^\circ + b)$

32 6; $\dfrac{\sqrt{3}}{\cos 2b} = \tan(60^\circ \pm b) + \tan(60^\circ \mp b)$

33 7; $\dfrac{4\sin 2b}{1 - 4\sin^2 b} = \tan(60^\circ \pm b) - \tan(60^\circ \mp b)$

34 8; $\dfrac{2\sqrt{3}}{4\cos^2 b - 1} = \cot(60^\circ \mp b) + \cot(60^\circ \pm b)$

35 9; $\dfrac{4\sin 2b}{4\cos^2 b - 1} = \cot(60^\circ \mp b) - \cot(60^\circ \pm b)$

These formulæ will be more than sufficient to show the manner in which this investigation is performed. They are, besides, of little use at present, although they were employed, at least in part, in the first construction of trigonometric tables, before they were expressed in an analytic form.

CHAPTER X.

Elementary Considerations in relation to the Application of Trigonometric Functions to Analysis, and to Calculations in general.

§ 38. It has been seen, by the series E, that the several trigonometric functions assume, in succession, every value between 0, and Infinity, both with the positive and negative sign. They are, in consequence, capable of representing every possible quantity that can occur in calculation, and the different combinations of the trigonometric functions give the same combinations of these quantities, that they do of the trigonometric functions themselves.

The inspection of several of these formulæ has already shown, that they may, for instance, serve to change an addition or subtraction into a multiplication or division; and thus transform a calculation by natural numbers, into one by logarithms; and in like manner to produce other changes of the form of calculations. Tables have even been made to facilitate such calculations. In the course of the solutions of triangles, whether plane or spheric, a frequent use will be made of them, by means of what are called auxiliary angles; and since the method is the same for all other quantities, it will be sufficiently illustrated by this application of it.

§ 39. There is another case where these functions are of great and general use in analysis. It deserves particular consideration, in consequence of the nature of the changes it demands in the trigonometric formulæ, and of its great utility. We speak of its application to those quantities that are called transcendental, which are reducible to circular arcs, and occur frequently in the integral calculus.

In order to prepare the way for this application of our formulæ, it must be, in the first place, observed: that if

$x = \text{sine } a$, we have $a = \text{arc whose sine is } x$. This is commonly expressed thus:

$$a = \text{arc} : \sin (x)$$

And so in all other cases; this expression is nothing more than the algebraic mode of expressing the idea; as in the case of sine, cosine, &c. If, then, we assume:

$$x = \sin a$$
$$y = \cos a$$
$$z = \tan a$$

we obtain for the elementary expressions of this mode of notation the following formulæ, which are analogous to those
T of series A in the beginning, viz:

1 $a = \text{arc} : \sin (x)$

2 $= \text{arc} : \cos (y)$

3 $= \text{arc} : \tan (z)$

4 $= \text{arc} : \cot \left(\frac{1}{z}\right)$

5 $= \text{arc} : \sec \left(\frac{1}{y}\right)$

6 $= \text{arc} : \text{cosec} \left(\frac{1}{x}\right)$

7 $= \text{arc} : \tan \left(\frac{x}{y}\right)$

8 $= \text{arc} : \cot \left(\frac{y}{x}\right)$

§ 40. We have also, by analogy with series C, the following properties of these same quantities, with all their possible algebraic variations; viz:

$$x^2 + y^2 = 1$$

$$1 + z^2 = \frac{1}{y^2}$$

$$1 + \frac{1}{z^2} = \frac{1}{x^2}$$

Whence may be deduced, as in the preceding section, the following formulæ:

$$a = \text{arc} : \sin\left(\sqrt{(1 - y^2)}\right) \qquad 9$$

$$= \text{arc} : \cos\left(\sqrt{(1 - x^2)}\right) \qquad 10$$

$$= \text{arc} : \tan\left(\frac{\sqrt{(1 - y^2)}}{y}\right) = \text{arc} : \tan\left(\frac{x}{\sqrt{(1 - x^2)}}\right) \qquad 11$$

$$= \text{arc} : \cot\left(\frac{\sqrt{(1 - x^2)}}{x}\right) = \text{arc} : \cot\left(\frac{y}{\sqrt{(1 - y^2)}}\right) \qquad 12$$

$$= \text{arc} : \sec\left(\frac{1}{\sqrt{(1 - x^2)}}\right) = \text{arc} : \sec\left(\sqrt{(1 + z^2)}\right) \qquad 13$$

$$= \text{arc} : \text{cosec}\left(\frac{1}{\sqrt{(1 - y^2)}}\right) = \text{arc} : \text{cosec}\left(\frac{\sqrt{(1 + z^2)}}{z}\right) \qquad 14$$

By means of these formulæ, therefore, we may express, by one or more arcs, any quantity that is given in the form of one of the trigonometric formulæ. Considering this quantity as representing the corresponding trigonometric function, the arc (here called a) will be that which is denoted by the trigonometric function, to which this quantity has been assumed to be equal, in conformity with the fundamental denominations assumed.

It must be also observed; that the simple function that is to be represented, must be that which corresponds to the function involved in the complex formula; since the object of this kind of transformation is always that of disengaging this quantity from its complications, by means of the relations of the several trigonometric functions; as in this example:

$$a = \text{arc} : \sin(x) = \text{arc} : \cot\left(\frac{\sqrt{(1 - x^2)}}{x}\right)$$

$$\begin{aligned} a &= \text{arc} : \cos\left(\sqrt{(1-x^2)}\right) \\ &= \text{arc} : \tan\left(\frac{x}{\sqrt{(1-z^2)}}\right) \\ &= \text{arc} : \sec\left(\frac{1}{\sqrt{(1-x^2)}}\right) \end{aligned}$$

and in like manner in every other case.

§ 41. To attain the object of this chapter, it will be sufficient to apply this mode of transforming functions to the fundamental formulæ of the series F and Q. These occur most frequently, and will show the manner of applying this method to every other form furnished by Analytic Trigonometry.

For the series F, we must also assume another arc $= b$; of which the functions, corresponding to those of the arc a, are best distinguished merely by an accent; so that we have for the two arcs a, and b,

$$\begin{aligned} x &= \sin a & \qquad x' &= \sin b \\ y &= \cos a & \qquad y' &= \cos b \\ z &= \tan a & \qquad z' &= \tan b \end{aligned}$$

with all their consequences, as above explained.

Substituting these values, the formula F, No. 1, will be represented in the following manner.

$$\begin{aligned} \sin(a \pm b) = x\,y' \pm x'\,y &= x\sqrt{(1-x'^2)} \pm x'\sqrt{(1-x^2)} \\ &= y'\sqrt{(1-y^2)} \pm y\sqrt{(1-y'^2)} \end{aligned}$$

Which will give, according to the principles that have been laid down, values such as:

$$\sin\left(\text{arc} : \sin(x) \pm \text{arc} : \sin(x')\right) = x\sqrt{\left(1-(x')^2\right)} \pm x'\sqrt{(1-x^2)}$$

and all the similar ones, that may be drawn from the preceding expressions.

If we take, on both sides of the equations, the arcs whose trigonometric functions are represented by these quantities, we obtain the result that is sought, viz:

U

$$\text{arc} : \sin(x) \pm \text{arc} : \sin(x')$$
$$= \text{arc} : \sin\left(x\sqrt{(1-x'^2)} \pm x'\sqrt{(1-x^2)}\right) \qquad 1$$

$$\text{arc} : \cos(y) \pm \text{arc} : \cos(y')$$
$$= \text{arc} : \sin\left(y'\sqrt{(1-y^2)} \pm y\sqrt{(1-y'^2)}\right) \qquad 2$$

From the formula for the cosine, F, No. 2, we obtain by a similar process:

$$\cos(a \pm b) = yy' \mp xx' = yy' \mp \sqrt{(1-y^2)}\sqrt{(1-y'^2)}$$
$$= \sqrt{(1-x^2)}\sqrt{(1-x'^2)} \mp xx'$$

Whence

$$\cos\left(\text{arc} : \sin(x) \pm \text{arc} : \sin(x')\right) = \sqrt{(1-x^2)}\sqrt{(1-x'^2)} \mp xx'$$
$$\cos\left(\text{arc} : \cos(y) \pm \text{arc} : \cos(y')\right) = yy' \mp \sqrt{(1-y^2)}\sqrt{(1-y'^2)}$$

And as final results:

$$\text{arc} : \sin(x) \pm \text{arc} : \sin(x')$$
$$= \text{arc} : \cos\left(\sqrt{(1-x^2)}\sqrt{(1-x'^2)} \pm xx'\right) \qquad 3$$

$$\text{arc} : \cos(y) \pm \text{arc} : \cos(y')$$
$$= \text{arc} : \cos\left(yy' \mp \sqrt{(1-y^2)}\sqrt{(1-y'^2)}\right) \qquad 4$$

The formula F, No. 4, gives:

$$\tan(a \pm b) = \frac{1 \pm z'\dfrac{1}{z}}{\dfrac{1}{z} \mp z'} = \frac{z \pm z'}{1 \mp zz'}$$

(The three next formulæ give forms that are identical.)
We obtain from this last expression:

$$\tan\left(\text{arc} : \tan(z) \pm \text{arc} : \tan(z')\right) = \frac{z \pm z'}{1 \mp zz'}$$

And finally:

$$\text{arc} : \tan(z) \pm \text{arc} : \tan(z') = \text{arc} : \tan\left(\frac{z \pm z'}{1 \mp zz'}\right) \qquad 5$$

It will be readily conceived, that the use of these expressions, combined in every possible manner, and even the introduction of the values of the tangents, instead of the values of the sines and cosines, and conversely, would furnish a multiplicity of formulæ; but they would become either complicated or identical; for it must be considered, that we do not treat of the quantities themselves, but of the form of their combinations.

It will be seen, that in this point of view, the formulæ U, 1 and 2, are already identical; for each of them shows, in each of its terms, a simple quantity, and the square root of Unity diminished by a square.

This is not the case in the formulæ deduced from the cosine; for in them the different products have different signs. The cotangents will evidently give the inverse of the tangents; and the same is the case with the secant and cosecant, in relation to the cosines and the sines.

§ 42. Applying this process to the series Q, we obtain:

By Q, No. 1; $\sin 2a = 2xy = 2x\sqrt{(1-x^2)} = 2y\sqrt{(1-y^2)}$

which gives

$$\sin\big(2 \text{ arc} : \sin(x)\big) = 2x\sqrt{(1-x^2)}$$

whence

$$6\left\{\begin{array}{l}\text{arc} : \sin(x) = \tfrac{1}{2}\text{ arc} : \sin\big(2x\sqrt{(1-x^2)}\big)\\ \text{and}\\ \text{arc} : \cos(y) = \tfrac{1}{2}\text{ arc} : \sin\big(2y\sqrt{(1-y^2)}\big)\end{array}\right.$$

By Q, No. 3 & 4;

$$\begin{aligned}\cos 2a &= 1-2x^2 = 1-2\sqrt{(1-y^2)}\\ &= 2\sqrt{(1-x^2)}-1 = 2y^2-1\\ \cos\big(2 \text{ arc} : \sin(x)\big) &= 1-2x^2 = 2\sqrt{(1-x^2)}-1\\ \cos\big(2 \text{ arc} : \cos(y)\big) &= 1-2\sqrt{(1-y^2)} = 2y^2-1\end{aligned}$$

and from these we obtain

$$7 \qquad \begin{aligned}\text{arc} : \sin(x) &= \tfrac{1}{2}\text{ arc} : \cos(1-2x^2)\\ &= \tfrac{1}{2}\text{ arc} : \cos\big(2\sqrt{(1-x^2)}-1\big)\end{aligned}$$

$$\text{arc} : \cos (y) = \tfrac{1}{2} \text{arc} : \cos \left(1 - 2 \sqrt{(1 - y^2)}\right)$$
$$= \tfrac{1}{2} \text{arc} : \cos (2y^2 - 1) \qquad 8$$

By Q, No. 5; $\tan 2a = \dfrac{2}{\dfrac{1}{z} - z} = \dfrac{2z}{1 - z^2}$

whence

$$\tan \left(2 \text{ arc} : \tan (z)\right) = \frac{2z}{1 - z^2}$$

$$\text{arc} : \tan z = \tfrac{1}{2} \text{arc} : \tan \left(\frac{2z}{1 - z^2}\right) \qquad 9$$

These formulæ, which may besides be useful, will suffice to give an idea of this application of Trigonometry in the analysis of infinitesimals.

◆

CHAPTER XI.

General Formulæ for the Trigonometric Functions of Multiple Angles, in Terms of the Functions of the Simple Angle only.

§ 43. In chapter VII. we have exhibited general and simple formulæ for the sine, cosine, and tangent, of a multiple angle, in terms of the functions of its aliquot parts; and likewise their application to successive multiples, in the series P.

By successive substitutions and reductions, we might deduce from these, expressions in terms of the functions of the simple angle only.

For the sake of greater simplicity, we shall here deduce these formulæ from those of series F, No. 1 and 2, alone. After a small number of multiples have been examined, a general law will be discovered, by which the determination of the numerical coefficients, and the order in which the powers of the several functions succeed each other, is prescribed. This law will thus be established by induction; and in the succeeding chapter we shall apply it, in order to give an idea of the manner in which trigonometric tables may be constructed; a subject whose principles must be understood, although it cannot be here treated of in all its details.

It may easily be seen, that a variety of formulæ might be obtained, but we shall treat of those only which are the most simple.

§ 44. If, in the formulæ F, No. 1 and 2, we assume $a = b$, we have seen in section 34, by the formulæ Q, No. 1 and 2, that:

$$\sin 2a = 2 \sin a \cos a$$

$$\cos 2a = \cos^2 a - \sin^2 a$$

Continuing this process, by means of successive assumptions, such as

$$b = 2a; \quad \text{whence} \quad a + b = 3a$$

we obtain for this case in the first place, by F, No. 1:

$$\sin 3a = \sin 2a \cos a + \cos 2a \sin a$$

and

$$\cos 3a = \cos 2a \cos a - \sin 2a \sin a$$

Substituting the values of sine $2a$, and cosine $2a$, in conformity to the value just preceding, we obtain:

$$\begin{aligned} \sin 3a &= 2 \sin a \cos^2 a + \sin a \cos^2 a - \sin^3 a \\ &= 3 \sin a \cos^2 a - \sin^3 a \end{aligned}$$

and

$$\begin{aligned} \cos 3a &= \cos^3 a - \cos a \sin^2 a - 2 \sin^2 a \cos a \\ &= \cos^3 a - 3 \sin^2 a \cos a \end{aligned}$$

For a quadruple angle we shall have:

$$\sin 4a = \sin 3a \cos a + \cos 3a \sin a$$
$$\cos 4a = \cos 3a \cos a - \sin 3a \sin a$$

And substituting from the last:

$$\sin 4a = 3 \sin a \cos^3 a - \sin^3 a \cos a + \cos^3 a \sin a - 3 \sin^3 a \cos a$$

Reducing:

$$\sin 4a = 4 \sin a \cos^3 a - 4 \sin^3 a \cos a$$

in like manner:

$$\cos 4a = \cos^4 a - 3 \sin^2 a \cos^2 a - 3 \cos^2 a \sin^2 a + \sin^4 a$$
$$= \cos^4 a - 6 \sin^2 a \cos^2 a + \sin^4 a$$

We obtain for a quintuple angle the following results in succession:

$$\sin 5a = \sin 4a \cos a + \cos 4a \sin a$$
$$= 4 \sin a \cos^4 a - 4 \sin^3 a \cos^2 a + \sin a \cos^4 a - 6 \sin^3 a \cos^2 a + \sin^4 a$$
$$= 5 \sin a \cos^4 a - 10 \sin^3 a \cos^2 a + \sin^5 a$$

and

$$\cos 5a = \cos 4a \cos a - \sin 4a \sin a$$
$$= \cos^5 a - 6 \sin^2 a \cos^3 a + \sin^4 a \cos a - 4 \sin^2 a \cos^3 a + 4 \sin^4 a \cos a$$
$$= \cos^5 a - 10 \sin^2 a \cos^3 a + 5 \sin^4 a \cos a$$

It is easy to extend this calculation to the subsequent multiples, for which reason we shall only give the results.

$$\sin 6a = 6 \sin a \cos^5 a - 20 \sin^3 a \cos^3 a + 6 \sin^5 a \cos a$$
$$\cos 6a = \cos^6 a - 15 \sin^2 a \cos^4 a + 15 \sin^4 a \cos^2 a - \sin^6 a$$
$$\sin 7a = 7 \sin a \cos^6 a - 35 \sin^3 a \cos^4 a + 21 \sin^5 a \cos^2 a - \sin^7 a$$
$$\cos 7a = \cos^7 a - 21 \cos^5 a \sin^2 a + 35 \cos^3 a \sin^4 a - 7 \cos a \sin^6 a$$

The simplest, and therefore the best, mode of considering the tangent, is, by dividing the sine by the cosine, and reducing. By this process we have:

$$\tan 2a = \frac{\sin 2a}{\cos 2a} = \frac{2 \sin a \cos a}{\cos^2 a - \sin^2 a} = \frac{2 \tan a}{1 - \tan^2 a}$$

$$\tan 3a = \frac{3 \sin a \cos^2 a - \sin^3 a}{\cos^3 a - 3 \sin^2 a \cos a} = \frac{3 \tan a - \tan^3 a}{1 - 3 \tan^2 a}$$

To obtain the second result, we divide constantly by that power of the cosine which constitutes the first term of the denominator.

It may be proper, before proceeding farther, to unite all these results in a table; by which their use may be facilitated, in obtaining conclusions from them for the general formulæ that are the object of this investigation.

W

$$1 \left\{ \begin{aligned} \sin a &= \sin a \\ \sin 2a &= 2 \sin a \cos a \\ \sin 3a &= 3 \sin a \cos^2 a - \sin^3 a \\ \sin 4a &= 4 \sin a \cos^3 a - 4 \sin^3 a \cos a \\ \sin 5a &= 5 \sin a \cos^4 a - 10 \sin^3 a \cos^2 a + \sin^5 a \\ \sin 6a &= 6 \sin a \cos^5 a - 20 \sin^3 a \cos^3 a + 6 \sin^5 a \cos a \\ \sin 7a &= 7 \sin a \cos^6 a - 35 \sin^3 a \cos^4 a + 21 \sin^5 a \cos^2 a - \sin^7 a \\ \sin 8a &= \&c. \end{aligned} \right.$$

$$2 \left\{ \begin{aligned} \cos a &= \cos a \\ \cos 2a &= \cos^2 a - \sin^2 a \\ \cos 3a &= \cos^3 a - 3 \sin^2 a \cos a \\ \cos 4a &= \cos^4 a - 6 \sin^2 a \cos^2 a + \sin^4 a \\ \cos 5a &= \cos^5 a - 10 \sin^2 a \cos^3 a + 5 \sin^4 a \cos a \\ \cos 6a &= \cos^6 a - 15 \sin^2 a \cos^4 a + 15 \sin^4 a \cos^2 a - \sin^6 a \\ \cos 7a &= \cos^7 a - 21 \sin^2 a \cos^5 a + 35 \sin^4 a \cos^3 a - 7 \sin^6 a \cos a \\ \cos 8a &= \&c. \end{aligned} \right.$$

$$\left.\begin{array}{lcl}
\tan a & = & \tan a \\
\tan 2a & = & \dfrac{2 \tan a}{1 - \tan^2 a} \\
\tan 3a & = & \dfrac{3 \tan a - \tan^3 a}{1 - 3 \tan^2 a} \\
\tan 4a & = & \dfrac{4 \tan a - 4 \tan^3 a}{1 - 6 \tan^2 a + \tan^4 a} \\
\tan 5a & = & \dfrac{5 \tan a - 10 \tan^3 a + \tan^5 a}{1 - 10 \tan^2 a + 5 \tan^4 a} \\
\tan 6a & = & \dfrac{6 \tan a - 20 \tan^3 a + 6 \tan^5 a}{1 - 15 \tan^2 a + 15 \tan^4 a - \tan^6 a} \\
\tan 7a & = & \dfrac{7 \tan a - 35 \tan^3 a + 21 \tan^5 \mathrm{a} - \tan^7 a}{1 - 21 \tan^2 a + 35 \tan^4 a - 7 \tan^6 \mathrm{a}} \\
\tan 8a & = & \&c.
\end{array}\right\} 3$$

§ 45. If we consider the foregoing formulæ for the sine and cosine of the multiple angles expressed wholly in terms of the sines and cosines of the simple angles, and their successive powers, both in relation to the order in which these powers, and to that in which their coefficients, occur, we shall perceive, that: for every corresponding multiple of the sine and cosine, beginning at the first term of the cosine, thence passing to the first term of the sine, then from the second term of the cosine to the second of the sine, and so on to the end; we have all the terms of the binomial in regular order, as well for the powers of cosine *a*, and sine *a*, as for their numeric coefficients; with this difference only, that a regular change of the signs, +, and —, takes place separately, in each of the series.

The same law holds good in the case of the tangents, as far as regards the coefficients; and the powers of the tangents follow in a regular order, from the numerator to the denominator, alternately.

§ 46. We may, therefore, substitute the terms of the binomial theorem in the formulæ, which will express at one view the general law. Calling, then, the number that expresses the multiple of the angle, n, we shall have the following general formulæ, viz:

$$
\begin{aligned}
\sin na = {} & n \sin a \cos^{n-1} a - \frac{n(n-1)(n-2)}{1\,.\,2\,.\,3} \sin^3 a \cos^{n-3} a \\
& + \frac{n(n-1)(n-2)(n-3)(n-4)}{1\,.\,2\,.\,3\,.\,4\,.\,5} \sin^5 a \cos^{n-5} a \\
& - \frac{n(n-1)\ldots\ldots(n-6)}{1\,.\,2\ldots\ldots\ldots\ldots 7} \sin^7 a \cos^{n-7} a
\end{aligned} \tag{4}
$$

$$
\begin{aligned}
\cos n a = {} & \cos^n a - \frac{n(n-1)}{1\,.\,2} \cos^{n-2} a \sin^2 a \\
& + \frac{n(n-1)(n-2)(n-3)}{1\,.\,2\,.\,3\,.\,4} \cos^{n-4} a \sin^4 a \\
& - \frac{n(n-1)\ldots\ldots(n-5)}{1\,.\,2\ldots\ldots\ldots\ldots 6} \cos^{n-6} a \sin^6 a \\
& + \frac{n(n-1)\ldots\ldots(n-7)}{1\,.\,2\ldots\ldots\ldots\ldots 8} \cos^{n-8} a \sin^8 a - \&c. + \&c.
\end{aligned} \tag{5}
$$

Making, in these two series, cosine^n a, a common factor to the whole series, they present series with the powers of the tangent of the simple arc in regular succession; thus:

$$
\begin{aligned}
\sin n a = \cos^n a \Big(& n \tan a - \frac{n(n-1)(n-2)}{2\,.\,3} \tan^3 a \\
& + \frac{n(n-1)(n-2)(n-3)(n-4)}{2\,.\,3\,.\,4\,.\,5} \tan^5 a \\
& - \frac{n(n-1)(n-2)\ldots\ldots(n-6)}{2\,.\,3\,.\,4\,.\,5\,.\,6\,.\,7} \tan^7 a \\
& + \frac{n(n-1)(n-2)\ldots\ldots(n-8)}{2\,.\,3\,.\,4\,.\,5\,.\,6\,.\,7\,.\,8\,.\,9} \tan^9 a - \&c. + \&c. \Big)
\end{aligned} \tag{6}
$$

$$\cos n\,a = \cos^n a \left(1 - \frac{n\,(n-1)}{2}\tan^2 a + \frac{n\,(n-1)\,(n-2)\,(n-3)}{2\,.\,3\,.\,4}\tan^4 a - \frac{n\,(n-1)\,(n-2)\,(n-3)\,(n-4)\,(n-5)}{2\,.\,3\,.\,4\,.\,5\,.\,6}\tan^6 a + \frac{n\,(n-1)\,(n-2)\,(n-3)\ldots\ldots(n-7)}{2\,.\,3\,.\,4\,.\,5\,.\,6\,.\,7\,.\,8}\tan^8 a - \frac{n\,(n-1)\ldots\ldots(n-9)}{2\ldots\ldots\ldots 10}\tan^{10} a + \&c. - \&c.\right) \qquad 7$$

By the division of 6 by 7, the series for the tangent becomes:

$$\tan na = \frac{n\tan a - \frac{n(n-1)\,(n-2)}{2\,.\,3}\tan^3 a + \frac{n\ldots\ldots(n-4)}{1\ldots\ldots\ldots 5}\tan^5 a - \frac{n\,(n-1)\ldots\ldots(n-6)}{2\ldots\ldots\ldots 7}\tan^7 a + \&c.}{1 - \frac{n(n-1)}{2}\tan^2 a + \frac{n\ldots\ldots(n-3)}{1\ldots\ldots\ldots 4}\tan^4 a - \frac{n\ldots\ldots(n-5)}{1\ldots\ldots\ldots 6}\tan^6 a + \frac{n\,(n-1)\ldots\ldots(n-7)}{2\ldots\ldots\ldots 8}\tan^8 a - \&c.} \qquad 8$$

Performing the division which is here indicated, (which may be done most easily by the method of indeterminate coefficients, that will be explained hereafter,) it is immediately discovered, that every subsequent term of the resulting series depends on all the preceding ones, by a combination of the binomial coefficients; the law of which is easy and simple, although the series itself, when developed, becomes long and complicated, though regular. It shall be represented here in the first shape. For this purpose, let the series be represented by the following, with undetermined coefficients, which, it will be seen, may all be determined from each other when the first is determinate, as is the case.

$$\begin{aligned}\tan n\,a &= A \tan a + B \tan^3 a + C \tan^5 a + D \tan^7 a \\ &\qquad + E \tan^9 a + F \tan^{11} a\end{aligned}$$

The determination of these coefficients gives $A = n$; and therefore, for the convenience of writing the result, denote the successive binomial coefficients thus; call:

$$\frac{n-1}{2} = n_1\ ; \quad \frac{n-2}{3} = n_2\ ; \quad \frac{n-3}{4} = n_3\ ;$$

$$\frac{n-4}{5} = n_4\ ; \quad \frac{n-5}{6} = n_5\ ; \quad \&c.$$

Then will the series be represented in the following form:

$$9\left\{\begin{array}{l} \tan n\,a = n \tan a + (n\, n_1\, A - n\, n_1\, n_2) \tan^3 a \\ \quad + (n\, n_1\, B - n\, n_1\, n_2\, n_3\, A + n\, n_1\, n_2\, n_3\, n_4) \tan^5 a \\ \quad + (n\, n_1\, C - n\, n_1\, n_2\, n_3\, B + n\, n_1 \ldots n_5\, A - n\, n_1 \ldots n_6) \tan^7 a \\ \quad + (n\, n_1\, D - n\, n_1\, n_2\, n_8\, C + n \ldots n_5\, B - n \ldots n_7\, A \\ \quad + n \ldots n_6) \tan^9 a + (n\, n_1\, E - n\, n_1\, n_2\, n_3\, D \\ \quad + n \ldots n_5\, C - n \ldots n_7\, B + n \ldots n_9\, A - n \ldots n_8) \tan^{11} a + \&c. \end{array}\right.$$

which may evidently be continued with ease, as the law is apparent. The developement of the factor remaining somewhat complicated, it may be omitted here, particularly as in all calculations of series, the terms are calculated in succession, the first term being always made the largest.

CHAPTER XII.

Elementary Ideas relating to the Construction of Trigonometric Tables.

§ 47. We have already seen, that the circumference of the circle and the trigonometric functions are incommensurable; the latter cannot, therefore, be expressed in parts of the first, and conversely, except by approximation, or by the transcendental or infinitessimal analysis. The first of these methods was employed at first, before the latter had cleared an easy way to results of this nature.

In order, then, to give an idea of the methods that may be employed to determine the trigonometric functions, by the methods of the infinitesimal calculus, it is necessary that we previously give an idea of the form of this calculus.

In the series of formulæ E, we have already seen the expression $\frac{1}{0}$ = Infinity. By it we are to understand, that the quantity it expresses is greater than can be expressed by numbers, in the same way that 0 represents the absence of all quantity. An attentive observation of the values given by the series E, shows the complete circle of all possible quantities; for in it the ratios between the several lines are seen to increase from 0 to ∞, both positive and negative; and undergoing changes of sign in both transitions, through these extreme values. This shows, at the same time, that all consideration of quantity is wholly relative, for it is from a ratio that this result is obtained; a result, as simple as it is valuable, in mathematical researches.

We ground upon the foregoing, a principle of constant use in the infinitesimal calculus; it consists in repelling or rejecting in this calculus every quantity which is not multiplied by ∞, as not appertaining to the hypothesis on which this calculus is founded.

Admitting then ∞ into the series of symbols, that express quantity, we shall have $\frac{1}{\infty}$ equal to what is called: infinitely small, and as in every species of calculus we must employ the conventional signs in conformity with their conventional signification, and the value attributed to them, we have

$$\infty\frac{1}{\infty} = 1; \text{ or } \frac{a\,\infty}{\infty} = a;$$

that is to say, a quantity divided by ∞, and multiplied by ∞, is equal to the quantity itself, as is the case with any other number; all this is no more than a form of calculus, by which the properties of ∞ are determined, or agreed upon, as is the case with all other symbols that represent quantity. We must never lose sight of the principle, that in analysis in general we only consider the form of the combinations of quantities, without regard to the quantities themselves; except so far as regards their ratios to certain other quantities, that are compared or placed in relation with them.

When what has been said above, is once understood, it will be easy to comprehend the use that is made of these principles, in this chapter; in which we show the manner of determining, by means of the general formulæ already given, the trignometric functions of a given angle or arc, upon the assumption, that the value of π is known; which supposition is then justified by the inversion of these results, so as to determine the value of π, by means of the preceding formulæ, expressed in parts of the radius taken equal to Unity.

§ 48. In conformity with the principles just laid down, it will be sufficient to assume, in the series W, No. 4, 5, 6, and 7, the arc $a = \frac{1}{\infty}$; that is to say, infinitely small; and the number by which it is multiplied, $n = \infty$, and investigate the consequences of this hypothesis, in conformity with the principles and formulæ already explained.

It will be ascertained, by means of the values found in the series E, that when $a = \frac{1}{\infty}$; the value of $\cos a = 1$; the sine or the tangent is equal to the arc itself, being all perpendiculars at the end of the radius $= 1$, and indefinitely small.

This supposition will transform the formulæ quoted, into such as contain only the powers of the quantity, ($n\,a$,) and the numbers that are found in the denominators of the several terms. (To make this more clear, as well as for the sake of brevity, we shall here suppose $n\,a = x$.)

By this method of proceeding, the formula W, No. 4, is transformed thus:

$$\sin x = x - \frac{x^3}{2.3} + \frac{x^5}{2.3.4.5} - \frac{x^7}{2.3.4.8.6.7} + \frac{x^9}{2.4.5.6.7.8.9}$$

$$- \frac{x^{11}}{2.3.......10.11} + \&c. \qquad \mathrm{X}\ 1$$

and the formula W, No. 5, into the following:

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{2.3.4} - \frac{x^6}{2.3.4.5.6} + \frac{x^8}{2.3.4.5.6.7.8}$$

$$- \frac{x^{10}}{2.3.......9.10} + \frac{x^{12}}{2.3.......11.12} + \&c. \qquad 2$$

The formula for the tangent W, No. 9, becomes

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{3.5} + \frac{17x^7}{3^2.5.7} + \frac{2.31\,x^9}{3^2.5.7{\cdot}9} + \frac{2.691.x^{11}}{3^2.5^2.7.9.11}$$

$$+ \frac{2.8447.x^{13}}{3^3.5^2.7.9.11.13} + \&c. \qquad 3$$

§ 49. These series then give, by approximation, the sine, cosine, and tangent of an arc ($n\,a = x$) which is supposed to be given under the hypothesis, that $n = \infty$, and $a = \frac{1}{\infty}$; this arc being expressed in the same terms as π, or in terms of the radius.

The inspection of the formulæ that present increasing powers of this arc, shows, that in order to obtain a series whose following terms shall each be less than that which precedes it, (x,) must be a fraction, the powers of which constantly decrease; this will always be the case here, as the arc equal to the radius, which is the unity in which π is given, is more than 57°.17′.44″, 8, and we have seen, (in chapter 9,) that the trigonometric functions need at farthest be calculated to 45°. A multiplicity of formulæ give the values of the trigonometric functions of compound and multiple arcs, from the functions of their parts; it is therefore sufficient to calculate the latter, by those series, in a proper manner, to obtain all those which may be necessary.

The smaller the arc x, the fewer terms of the series will be needed, in order to obtain the value accurately to a given number of decimal places: but in this respect it is obviously necessary: that, as the result of the first calculation is to be multiplied, to obtain functions of the greater arcs, we must give to the first calculation or value, a proportionally greater number of decimals.

These explanations will suffice to give an idea of the manner in which trigonometric tables may be calculated, which is all that we need illustrate in this treatise.

§ 50. The last condition that remains, is, to determine the value of π, in terms of the diameter, or rather, of $\frac{1}{2}\pi$, in terms of the radius, which is evidently the same thing. The series which have been pointed out, perform this office also, by means of the process in calculation, called the inversion of series. This process has in itself no difficulty; it will be explained by the application which shall here be made of it, in relation to the last of the above series, which is chosen here, on account of its leading by the most converging series, to the end here proposed.

To do this, a series is supposed given in the form in which it may be always easily foreseen that it will assume, in which the coefficients are indeterminate, and become determined in the course of the process.

For this purpose we shall assume, in the present case,

$$x = A \tan x + B \tan^3 x + C \tan^5 x + D \tan^7 x + E \tan^9 x + \text{\&c.}$$

We shall next express the several values of tangent x, tangent ${}^3x$, &c. by the preceding series, X, No. 3, and its corresponding powers, using no more terms than are necessary to determine the law of the progression of the coefficients. The sum of all the terms, will evidently give a new value of x, and subtracting x from both sides of the equation, the value of the resulting series becomes $= 0$; from this it necessarily follows, that each sum of the terms of the various powers of tangent x will itself be $= 0$, since the equation must be true whatever be the value of tangent x. From this consequence are deduced as many equations as there are undetermined coefficients; these coefficients therefore become determinable in succession; the value of the entire series will, in consequence, be determined by the insertion of those values in their proper places.

The following is the process:

Expressing the several values of the powers of tangent x, by the multiplication of series No. 3, itself, we obtain, after multiplying each by its appropriate coefficient,

$$\begin{aligned}
A \tan x &= Ax + \frac{A}{3} x^3 + \frac{2A}{3.5} x^5 + \frac{17A}{3^2.5.7} x^7 + \frac{62A}{3^2.5.7.9} x^9 + \\
B \tan^3 x &= \phantom{Ax +{}} B x^3 + B x^5 + \frac{11B}{3.5} x^7 + \frac{88B}{3^3.7} x^9 + \\
C \tan^5 x &= + C x^5 + \frac{5C}{3} x^7 + \frac{16C}{3^2} x^9 +
\end{aligned}$$

$$D \tan^7 x \;=\; \qquad D\,x^7 + \frac{7D}{3}\,x^9 \;+$$

$$E \tan^9 x \;=\; \qquad + \;E\,x^9 \;+$$

$$F \tan^{11} x \;=\; \qquad +$$

&c. &c.

The series given above for x, becomes thus equal to the sum of all these terms, and subtracting x from both sides of the equation

$$0 = (A - 1)\,x + \left(\frac{A}{3} + B\right)x^3 + \left(\frac{2A}{3.5} + B + C\right)x^5$$

$$+ \left(\frac{17A}{3^2.5.7} + \frac{11B}{3.5} + \frac{5C}{3} + D\right)x^7$$

$$+ \left(\frac{62A}{3^3.5.7.9} + \frac{88B}{3^3.7} + \frac{16C}{3^2} + \frac{7D}{3} + E\right)x^9 + \&c.$$

We thus obtain, for the determination of these coefficients, the following successive equations and results :

$$A - 1 = 0 \qquad \text{whence} \qquad A = 1$$

$$\frac{A}{3} + B = 0 \qquad\qquad B = -\frac{1}{3}$$

$$\frac{2A}{3.5} + B + C = 0 \qquad\qquad C = +\frac{1}{5}$$

$$\frac{17A}{3^2.5.7} + \frac{11B}{3.5} + \frac{5C}{3} + D = 0 \qquad\qquad D = -\frac{1}{7}$$

$$\frac{62A}{3^2.5.7.9} + \frac{88B}{3^3.7} + \frac{16C}{3^2} + \frac{7D}{3} + E = 0 \qquad\qquad E = +\frac{1}{9}$$

&c. &c.

Substituting these values of the coefficients, in the series assumed for the value of x, we finally obtain

$$x = \tan x - \tfrac{1}{3}\tan^3 x + \tfrac{1}{5}\tan^5 x - \tfrac{1}{7}\tan^7 x + \tfrac{1}{9}\tan^9 x - \tfrac{1}{11}\tan^{11} x + \&c. \qquad 4$$

§ 51. To make use of this series for the purpose of determining the value of π, we may make use of several methods, drawn from the combination of different arcs, whose tangents are rational, or expressed in simple fractions. As an example we shall only choose the following:

Assuming the arc of 45°, whose tangent $= 1$, to be compounded of two other arcs, the tangent of one of which $\tan a = \frac{1}{2}$; we have by formula F, No. 6,

$$\tan(a + b) = 1 = \frac{\frac{1}{2} + \tan b}{1 - \frac{1}{2}\tan b}$$

which gives the value of

$$\tan b = \tfrac{1}{3}$$

We therefore have two tangents, expressed in simple fractions, that may be introduced into the series X, No. 4, and whose sum will give the value of the arc of 45°. Four times this value is the half circumference, or the value of π, which is the quantity sought, expressed in terms of the radius $= 1$. This substitution in the series gives

$$\pi = 4\left\{\begin{array}{l} \frac{1}{2} - \frac{1}{3}(\frac{1}{2})^3 + \frac{1}{5}(\frac{1}{2})^5 - \frac{1}{7}(\frac{1}{2})^7 + \frac{1}{9}(\frac{1}{2})^9 - \frac{1}{11}(\frac{1}{2})^{11} \\ \frac{1}{3} - \frac{1}{3}(\frac{1}{3})^3 + \frac{1}{5}(\frac{1}{3})^5 - \frac{1}{7}(\frac{1}{3})^7 + \frac{1}{9}(\frac{1}{3})^9 - \frac{1}{11}(\frac{1}{3})^{11} \\ \qquad + \frac{1}{13}(\frac{1}{2})^{13} - \frac{1}{15}(\frac{1}{2})^{15} + \&c. \\ \qquad + \frac{1}{13}(\frac{1}{3})^{13} - \frac{1}{15}(\frac{1}{3})^{15} + \&c. \end{array}\right\} \qquad 5$$

Performing these calculations to a sufficient number of terms, we obtain, in whole numbers and decimals,

$$\pi = 3{,}1415926535897932384626433832795, \&c. \qquad 6$$

This being the value of an arc of 180°, in terms of the radius, it may be seen how the value of any arc whatsoever, may be expressed in parts of the same radius, by taking a proportional part of this value of 180°. This value of an arc $= x$, may then be inserted in the series X, No. 1, 2, 3, to calculate from it the *sine*, *cosine*, or *tangent*, in conformity to the assumptions that have been made in speaking of these series.

§ 52. We have thus completed the circle of our analytic investigation, in relation to the trigonometric functions; beginning with the elementary definitions, or functions, deduced from the ratios existing between the sides of a right-angled plane triangle, and proceeding until we reach the determination of the value of the circumference of the circle in terms of the radius; which last value serves as a foundation for the calculation of trigonometric functions in actual numbers.

By means of series more advantageous than those which have been given, but whose investigation here would lead us too far, the number π, has been calculated to 148. places of decimals; an accuracy far beyond what is ever necessary in any calculation whatever; and which, even more than ever, renders it useless to search for the quadrature of the circle.

It cannot belong to this treatise to treat of logarithmic series for the trigonometric functions; logarithms in general, forming no part of our plan, would introduce a complication that is not intended. In the 4th part of this treatise, their knowledge is, however, necessarily supposed, at least so far as all trigonometric tables usually teach, in their introduction.

PART II.

PLANE TRIGONOMETRY.

CHAPTER I.

Solution of all the Cases of Oblique Angled Plane Trigonometry.

§ 53. Provided with the analysis, the results, and formulæ of the foregoing chapters, Oblique Angled Plane Trigonometry becomes an easy application of the formulæ we have obtained to the solution of all its several problems. We shall here treat of it in this point of view.

In order to determine a lineal dimension, it is necessary that one of the given quantities should be also a lineal dimension.

To determine the absolute value of a triangle, we must therefore have, among the data, one of its sides, for it is well known, by the elementary principles of geometry, that the determination of the angles only, determines nothing more than the similarity of the triangles.

We now give the various problems with their solutions.

§ 54. *Problem* 1. To find the relation between the sides and one of the trignometric functions of the angles of an oblique angled plane triangle.

Let ABC, (figure 6, and 7,) be an oblique angled plane triangle, whose sides are a, b, c, respectively opposite the angles of the same name. From the point A let fall the perpendicular $AD = d$ upon the opposite side $BC = a$, or upon that side produced, if the angle B, or C, be obtuse, (as in fig.

7,) the triangle ABC will furnish two right angled triangles, ABD, and ACD, having the side, $AD = d$, common.

Solution. By the formula No. 1, of the series A, or first definition, we have in the two triangles, and in both cases, (since the sine of an angle is equal to the sine of its supplement.)

$$\frac{d}{b} = \sin C \;; \quad \text{and} \quad \frac{d}{c} = \sin B$$

Therefore:

$$d = b \,.\, \sin C = c \,.\, \sin B$$

Or, expressed in a proportion:

Y

$$b : c = \sin B : \sin C$$

And this being general for any side, we have also:

1

$$b : a = \sin B : \sin A$$

$$a : c = \sin A : \sin C$$

This is generally expressed thus:

In any plane triangle, the sides are to each other as the sines of their opposite angles. It gives therefore the solution of all the cases, where two of the parts given are opposite to each other, and the part to be found opposite to its corresponding given part.

Corollary. If from the three angular points of a triangle, perpendiculars be let fall upon the opposite sides, these perpendiculars are to each other in the inverse ratio of the sides on which they fall.

In the triangle ABC, (fig. 8,) let d, d', be the perpendiculars falling upon the sides a, and b, respectively; we have by A, No. 1, as before,

$$\sin C = \frac{d}{b} = \frac{d'}{a} \;; \quad \text{or} \quad d : d' = b : a$$

§ 55. *Problem* 2. In an oblique angled plane triangle, given the two sides and the included angle, to find the two remaining angles.

The sum of the three angles of a plane triangle is always equal to two right angles; (elementary geometry;) in this case then we have given not only the two sides, and the angle included, but also the sum of the two angles sought; all then that is necessary to determine each angle separately, is to find their difference; for the largest is equal to half the sum increased by half the difference, and the least to half the sum diminished by half the difference.

In the same triangle that has been used in the first problem, having given A, b, and c, we have, as has been demonstrated,

$$b : c = \sin B : \sin C$$

And by composition of this proportion,

$$b + c : b - c = \sin B + \sin C : \sin B - \sin C$$

Substituting from series N, No. 3,

$$b + c : b - c = \tan \tfrac{1}{2} (B + C) : \tan \tfrac{1}{2} (B - C)$$

whence:

$$\tan \tfrac{1}{2} (B - C) = \tan \tfrac{1}{2} (B + C) \frac{b - c}{b + c}$$

And since the three angles, $A + B + C = 180°$,

or $\tfrac{1}{2} A + \tfrac{1}{2} (B + C) = 90°$,

we have: $90° - \tfrac{1}{2} A = \tfrac{1}{2} (B + C)$

and $\tan \tfrac{1}{2} (B + C) = \tan (90° - \tfrac{1}{2} A) = \cot \tfrac{1}{2} A$

The formula becomes:

$$\tan \tfrac{1}{2} (B - C) = \cot \tfrac{1}{2} A \frac{b - c}{b + c} \tag{2}$$

And calling $\dfrac{d}{2} = \dfrac{B - C}{2}$; we have the two angles, avoiding all unnecessary subtraction, (and B being considered the greater angle.)

$$B = 90^\circ - \tfrac{1}{2} A + \tfrac{1}{2} d$$
$$C = 90^\circ - \tfrac{1}{2} A - \tfrac{1}{2} d$$

This formula requires in its use, an addition and a subtraction. It may, when desired, be adapted to the calculation of quantities given in logarithms, a case that occurs in astronomy, by the following transformations.

For this purpose, we divide the numerator and denominator of the fractional part of the formula by b; and we have:

$$\tan \tfrac{1}{2} (B - C) = \cot \tfrac{1}{2} A \, \frac{1 - \dfrac{c}{b}}{1 + \dfrac{c}{b}}$$

Comparing this form of the fractional part with the formula S, No. 13, using the lower signs, it will be found: that,

3 if we call $\dfrac{c}{b}$ = tangent z, we have:

$$\tan (45^\circ - z) = \frac{1 - \tan z}{1 + \tan z}$$

Taking, then, for the calculation of the data given in logarithms, tangent $z = \dfrac{c}{b}$, we obtain for the solution of this case the formula:

4 $$\tan \tfrac{1}{2} (B - C) = \cot \tfrac{1}{2} A \tan (45^\circ - z)$$

This is an instance of the application of the preceding analytic formulæ for trigonometric functions, to the transformation of an expression containing addition and subtraction into one that can be calculated by logarithms alone; and we shall always have $\frac{c}{b} \angle 1$, according to the original supposition of $B \measuredangle C$.

§ 56. *Problem* 3. Given two sides and the included angle, to find the third side.

Given, a, c, and B, to find b; in the same triangle, let d denote the perpendicular upon a, as in the first problem.

By the elementary formulæ of series A, we have:

$$\frac{BD}{c} = \cos B \; ; \quad \text{thence} \quad BD = c.\cos B$$

and $$\frac{d}{c} = \sin B \; ; \qquad d = c.\sin B$$

and $$CD = a - BD = a - c.\cos B$$

By Geometry:

$$\begin{aligned} b^2 &= d^2 + CD^2 = c^2.\sin^2 B + (a - c.\cos B)^2 \\ &= c^2 \sin^2 B + a^2 + c^2 \cos^2 B - 2a.c.\cos B \\ &= c^2 (\sin^2 B + \cos^2 B) + a^2 - 2a.c \cos B \end{aligned}$$

And because $sine^2 + cosine^2 = 1$:

$$b^2 = c^2 + a^2 - 2a.c.\cos B \qquad 5$$

This formula, which would be very inconvenient to calculate, may be reduced to an easy form for logarithmic calculation, in two different ways. These will be obvious, when we consider that the formulæ Q, No. 4 and 3, give two different values for cosine B, the one in the sine, the other in the cosine, of the half angle.

Taking, then, in the first place:

$$\cos B = 1 - 2 \sin^2 \tfrac{1}{2} B$$

And substituting this value in the equation for cosine B, we have:

$$\begin{aligned} b^2 &= c^2 + a^2 - 2ac + 4.c.a.\sin^2 \tfrac{1}{2} B \\ &= (a \backsim c)^2 + 4.a.c.\sin^2 \tfrac{1}{2} B \end{aligned}$$

And using $(a \backsim c)$ as a factor common to both the right hand terms:

$$b^2 = (a \backsim c)^2 \left(1 + \frac{4.c.a.\sin^2 \frac{1}{2} B}{(a \backsim c)}\right).$$

Extracting the root:

6 $$b = (a \backsim c) \left(1 + \frac{4.c.a.\sin^2 \frac{1}{2} B}{(a \backsim c)^2}\right)^{\frac{1}{2}}.$$

It may be seen: that, by the method used in the preceding Problem, this formula may be adapted to the use of logarithms, employing the consideration: that, according to C, No. 8, we have:

$$\sec^2 = 1 + \tan^2 = \frac{1}{\cos^2}$$

If, then, we make

$$\tan^2 x = \frac{4.a.c.\sin^2 \frac{1}{2} B}{(a \backsim c)^2}$$

7 or $$\tan x = \frac{2 \sin \frac{1}{2} B}{a \backsim c} \sqrt{(a.c)}$$

the final formula will become:

8 $$b = \frac{a \backsim c}{\cos x}$$

Second Transformation. Taking;

$$\cos B = 2 \cos^2 \tfrac{1}{2} B - 1$$

and substituting this value in No. 5, the formula becomes:

$$b^2 = a^2 + c^2 + 2ac - 4.a.c.\cos^2 \tfrac{1}{2} B$$

And by processes exactly similar to those used for the formula 6, it will finally become:

$$b = (a + c)\left(1 - \frac{4.c.a.\cos\frac{1}{2}B}{(a + c)^2}\right)^{\frac{1}{2}} \qquad 9$$

Here we have, by the same means as in the foregoing transformation, representing the second term under the radical as the square of a sine or cosine, according to C, No. 4 or 5, and by an analogous process:

$$\cos^2 x = \frac{4.c.a.\cos^2\frac{1}{2}B}{(a + c)^2}$$

or

$$\cos x = \frac{2\cos\frac{1}{2}B}{a + c}(a.c)^{\frac{1}{2}} \qquad 10$$

And finally:

$$b = (c + a)\sin x \qquad 11$$

If the distances, a and c, were given in logarithms, (as may occur in astronomy,) it will be observed, that in applying to the part $(a \pm c)$, in the two formulæ 8 and 11, the same mode of transformation for the application of an auxiliary angle, by writing $(a \pm c) = a\left(1 \pm \frac{c}{a}\right)$ in both the places where it occurs, these formulæ will be transformed, and fitted for that use, upon the same principles, every thing else remaining as above.

§ 57. *Problem* 4. Given, the three sides of a plane triangle, to determine one of the angles, (suppose the angle B.)

Solution. The formula No. 5 gives this solution by simple transposition; for, from

$$b^2 = a^2 + c^2 - 2ac\cos B$$

it follows, that

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac} \qquad 12$$

This formula would also be inconvenient to calculate; it has the same two modes of transformation as the preceding, by the insertion of the sine or cosine of the half angle; and a third is obtained by the division of these two.

First Transformation. From the preparation for formula 6, is first obtained by transposition and division:

$$\sin^2 \tfrac{1}{2} B = \frac{b^2 - (a \backsim c)^2}{4.a.c}$$

The two terms of the numerator being the difference of two squares, the product of the sum and difference of their roots can be substituted for them; then, dividing the numerical coefficient of the denominator into two factors, and applying them to the two factors of the numerator, we obtain:

13 $$\sin^2 \tfrac{1}{2} B = \frac{\left(\frac{a + b - c}{2}\right)\left(\frac{b + c - a}{2}\right)}{a.c}$$

If we assume $p = \dfrac{a + b + c}{2}$, the expression becomes still simpler, for this assumption gives:

14 $$p - c = \frac{a + b - c}{2}\ ; \quad \text{and} \quad p - a = \frac{b + c - a}{2}$$

Inserting this in the formula, it becomes:

$$\sin^2 \tfrac{1}{2} B = \frac{(p - a)(p - c)}{a.c}$$

And extracting the root:

15 $$\sin \tfrac{1}{2} B = \left(\frac{(p - a)(p - c)}{a.c}\right)^{\frac{1}{2}}$$

Second Transformation. From the preparation for formula 9, we obtain, by transposition and division:

$$\cos^2 \tfrac{1}{2} B = \frac{(a + c)^2 - b^2}{4.a.c}$$

And by steps exactly analogous to the foregoing, and the supposition that $p = \dfrac{a + b + c}{2}$, as above, we obtain here:

$$\cos^2 \tfrac{1}{2} B = \frac{\left(\dfrac{a + c + b}{2}\right)\left(\dfrac{a + c - b}{2}\right)}{ac}$$

$$= \frac{p\,(p - b)}{ac}$$

$$\cos \tfrac{1}{2} B = \left(\frac{p\,(p - b)}{ac}\right)^{\frac{1}{2}} \qquad 16$$

When these formulæ are employed, we must choose that which gives the greatest degree of exactness in the given case. The principle which will determine this choice, is as follows: when the angle is small, the sine varies rapidly, while the cosine varies but little. In this case we should use the formula that employs sine ½ B; if the angle that is sought is obtuse, we must use the formula that gives the value of cosine ½ B, because it is the cosine that varies most in this case.

It will be observed, that, in general, the formulæ that determine the half angles have the advantage of avoiding all ambiguity between obtuse or acute angles; because they can never be obtuse, as in that case $B > 180°$, which is impossible.

We may deduce from these two formulæ a third, giving the tangent ½ B, that is applicable with equal advantage to all cases. For it is evident, that:

$$\frac{\sin \tfrac{1}{2} B}{\cos \tfrac{1}{2} B} = \tan \tfrac{1}{2} B = \left(\frac{(p - a)\,(p - c)}{p\,(p - b)}\right)^{\frac{1}{2}} \qquad 17$$

This requires two more preliminary steps to prepare the data for calculation; but they are short.

§ 58. *Problem 5.* Given, two sides, and an angle opposite to one of the sides; to find the third side.

C, *c*, and *b*, being given, to find *a*.

We find already, from inspection of *Fig.* 6 *and* 7, that the result may, in this case, be doubtful; as with these data, the angle at *B* may be either acute or obtuse, with *c* of equal magnitude, since this line is always greater than the perpendicular *d*, and, being drawn from the same point *A*, may cut *CB* on either side of this perpendicular; which will give two values of *CB*, that are equally possible, and both given by the formula. Circumstances, other than those of the mere magnitude of the parts given, will determine which of the two results is the true one.

Solution. In the formula Y, No. 4, *Problem* 3, we might suppose *b* given, and *a* to be determined, the other data remaining the same, and obtain this solution; this would lead to an equation of the second order. By the following process we reach the same result more easily.

By Y, No. 1, *problem* 1, we have:

$$c : b = \sin C : \sin B$$

whence

$$\sin B = \frac{b \sin C}{c}$$

From series A, No. 2, we have, by application of this case:

$$CD = b \cos C$$

and

$$BD = c \cos B$$

And according as *B* is acute or obtuse, we find:

$$BC = CD \pm BD = a$$

Expressing this in terms of the above two values, we have:

18 $$a = b \cos C \pm c \cos B$$

And substituting for cosine *B* its value deduced from the first enunciation above, expressing it, in conformity with series C, No. 5, the final result is:

19 $$a = b \cos C \pm c \left(1 - \frac{b^2 \sin^2 C}{c^2}\right)$$

A formula troublesome to calculate, though it may be reduced to logarithmic calculation, by the use of two auxiliary arcs; for we may take:

$$\sin x = \sin B = \frac{b \sin C}{c}$$

and thereby obtain:

$$\begin{aligned} a &= b \cos C \pm c \cos B \\ &= b \cos C \left(1 \pm \frac{c \cos B}{b \cos C}\right) \end{aligned}$$

where we are evidently again to assume, according to the case:

$$\left.\begin{matrix}\tan \\ \text{or } \sin\end{matrix}\right\} y = \left(\frac{c \cos B}{b \cos C}\right)^{\frac{1}{2}} \qquad 20$$

and would ultimately obtain:

$$a = \frac{b \cos C}{\cos^2 y} \qquad 21$$

$$\text{or} \qquad a = b \cos C \cos^2 y$$

But such a calculation would evidently be tedious, and it is far preferable to calculate sine B by *problem* 1, and then a by the two parts of the formula No. 18, above, unless it happen, that only the logarithms of b and c be given; when this mutation will be necessary and applicable.

§ 59. We have thus obtained the solutions of every possible case of Oblique Angled Plane Trigonometry, for data directly given, by means of formulæ affording the greatest facilities for calculation. It will be readily conceived: that what is said of an angle B, or of any side b, &c. is always to be understood, in a general sense, of every other angle or side. It is proper here, again to direct the attention to one general character of all the formulæ, namely: that they always present the linear dimensions in an even number of factors, and the trigonometric functions in an odd number of factors, when the part to be determined is a trigonometric function; and, conversely, the trigonometric functions in an even number of factors, and the linear dimensions in an odd number of factors, when a linear dimension is to be determined. This is the general and well known character of all proportions; from which, also, analytical formulæ cannot deviate, whatever be the complication of the data contained in them. The trigonometric functions being mere ratios, (that is, generally speaking, numbers,) as has been observed in section 11, they here, form as such, objects of a determined nature.

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CHAPTER II.

Calculation of the Surfaces of Plane Triangles from different Data.

§ 60. We think proper to add to the solutions of Trigonometry that give the unknown parts of triangles from those which are known, the solution of the problem: to find the surface of the triangle in every case of the before-mentioned data. This is evidently possible; for if we have the data necessary to determine the unknown parts of a triangle, the same data must also give its contents or surface. We shall take the cases in the same order as in the preceding solutions. It must, in the first place, be recollected, that the surface of the triangle is half the product of one of the sides, assumed as a base, into the perpendicular let fall upon it from the opposite angular point, as is taught in the elements of Geometry. It is, therefore, the principal object of these problems, to express the value of this perpendicular in terms of the parts given. Its product into the given base, divided by two, will, then, always give the surface, or contents of the triangle.

§ 61. *Problem* 1. Given, two angles and the included side; to find the Surface of the triangle.

In the triangle *ABC*, as before, let *BC* and *a* be given, to find the surface.

We have by Y, No. 1:

$$b = \frac{a \sin B}{\sin A} = \frac{a \sin B}{\sin (B + C)}$$

For the sine of an angle is equal to the sine of its supplement, not only in magnitude, but also in sign; and because the sum of all the three angles of a plane triangle is equal to two right angles, the sine of any one of the angles is equal to the sine of the sum of the two others.

Calling d the perpendicular let fall upon the side a, we have, as the value of this perpendicular:

$$d = b \sin C = \frac{a \sin B \sin C}{\sin (B + C)}$$

Multiplying this by half the base on which the perpendicular d falls, that is, $\frac{a}{2}$, and calling the surface of the triangle $= S$, we have:

$$S = \frac{a^2 \sin B \sin C}{2 \sin (B + C)} \qquad \text{Z} \; 1$$

And supposing the value of sine $(B + C)$ to be expressed in conformity to F, No. 1, and dividing both numerator and denominator by *sine B sine C*, we have also:

$$S = \frac{a^2}{2 (\cot B + \cot C)} \qquad 2$$

But the formula No. 1, is more convenient for logarithmic calculation.

§ 62. *Problem* 2. Given, two sides and the included angle; to find the Surface of the triangle.

Given, a, b, and C.

We have by the elementary definitions of series **A**, the value of the perpendicular:

$$d = b \,.\, \sin C$$

Multiplying this perpendicular by $\frac{a}{2}$, we have for the surface:

$$S = \frac{ab \sin C}{2}$$

§ 63. *Problem* 3. Given, the three sides; to find the Surface of the triangle.

By the theorem of geometry, so often employed: that the square of the hypothenuse is equal to the sum of the squares of the two sides, and expressing the parts by trigonometry, as in section 56, we find:

$$BD = c \,.\, \cos B$$

We have thence:

$$d^2 = c^2 - c^2 \cos^2 B$$

Then, expressing the value of c^2 cosine^2 B by the formula of section 57, and observing: that c^2, being found both in the numerator and denominator, compensates itself; we have:

$$d^2 = c^2 - \left(\frac{c^2 + a^2 - b^2}{2a}\right)^2$$

And thence:

$$d^2 = \frac{4\, a^2\, c^2 - (c^2 + a^2 - b^2)^2}{4\, a^2}$$

The two terms of the numerator, being the difference of the two squares, may be expressed by the product of the sum and difference of their roots; whence:

$$d^2 = \frac{(2\,ca + c^2 + a^2 - b^2)\,(2\,ca - c^2 - a^2 + b^2)}{4\,a^2}$$

$$= \frac{((c+a)^2 - b^2)\,(b^2 - (c \backsim a)^2)}{4a^2}$$

Applying the same principle again to the two factors of the numerator, we obtain:

$$d^2 = \frac{(c + a + b)\,(c + a - b)\,(c + b - a)\,(b + a - c)}{4\,a^2}$$

Then extracting the root:

$$d = \frac{1}{a}\left(\frac{(c + a + b)\,(c + a - b)\,(c + b - a)\,(b + a - c)}{4}\right)^{\frac{1}{2}}$$

This value of the perpendicular, being multiplied by half the base, $\frac{a}{2}$, gives the Surface:

$$S = \frac{da}{2}$$

$$S = \tfrac{1}{2}\left(\frac{(c + a + b)\,(c + a - b)\,(c + b - a)\,(b + a - c)}{4}\right)^{\frac{1}{2}}$$

Bringing the ½ under the radical, by squaring it, and distributing the fourth power of 2, which is produced by this insertion, among the four factors of the numerator, the final formula becomes:

$$S = \left(\frac{a + b + c}{2} \cdot \frac{c + a - b}{2} \cdot \frac{c + b - a}{2} \cdot \frac{a + b - c}{2}\right)^{\frac{1}{2}} \qquad 4$$

Which, by the introduction of $p = \dfrac{a+b+c}{2}$, as in section 57, becomes:

$$S = \left(p\,(p-a)\,(p-b)\,(p-c)\right)^{\frac{1}{2}} \qquad 5$$

§ 64. *Problem* 4. Given, two sides, and the angle opposite to one of them; to find the Surface.

Given, b, c, and C; to find S.

In figure 8, we have the perpendicular $BD' = a \text{ sine } C = d'$; and, according to general principles,

$$S = \frac{bd'}{2} = \frac{ba \cdot \sin C}{2}$$

Substituting for a the value given by section 58, formula 19, we obtain:

$$S = \frac{b^2 \cos C \sin C}{2} \pm \frac{c \,.\, b \,.\, \sin C}{2}\left(1 - \frac{b^2 \sin^2 C}{c^2}\right)^{\frac{1}{2}} \qquad 6$$

This formula evidently labours under the same disadvantages for calculation as Y, No. 19; therefore the remarks made there, equally apply here; that is, determining first B by Y, No. 1, and then S by Z, No. 1, will be preferable to this formula, even with the reductions that might be made in it.

§ 65. The four preceding solutions resolve the problem for every possible case of data; they complete the full series of solutions that may be required for a triangle with simple data.

It may be conceived, that it is possible, that, instead of the quantities themselves, it may happen that the sum and difference of some of them may be given, or some other relation between the parts. But it does not belong to an elementary system of Trigonometry to enter into these details. The principles here laid down show what parts must be determinable from the data, in order that a solution may be obtained; they also show, conversely, what data will serve to determine a triangle in complicated cases.

What remains to be said upon this subject, therefore, belongs properly to a further extension of the application of Trigonometry to the solutions of the problems that relate to triangles; and, consequently to all rectilineal figures, since Geometry teaches the means of decomposing them all into triangles.

It may, perhaps, be proper in this place to call the attention of the reader to the general character of the formulæ of series Z, analogous to what is said in section 59; namely: that they all present two lineal dimensions, with certain trigonometric functions as factors, which represent, as stated in section 11, mere numbers; that is, relations of quantities. This evidently characterizes them as giving in result a quantity of two dimensions, that is, a surface. It will easily be observed, that formula 4, having four lineal dimensions under the radical, presents also in its final result only two dimensions, by the extraction of the root.

PART III.

SPHERICAL TRIGONOMETRY.

CHAPTER I.

Introduction.—Lemmata, and Definitions of Spherical Trigonometry.

§ 66. WE know from the elements of Geometry: that the circle is generated by the revolution of a finite straight line around one of its extremities; and that the sphere is generated by the revolution of the circle around one of its diameters.

Then, because all the radii of the circle are equal, all the radii of the sphere are equal also, and all the circles that pass through the centre of the sphere, may be considered as generating circles; they are also the largest circles that can be described in a sphere, for which reason, they are called *Great Circles.*

§ 67. All the other circles of the sphere, that do not pass through its centre, are *Less Circles*, and are so called; they are also called *Parallels*, because they are necessarily parallel to some one great circle of the sphere; the consideration of their properties forms no part of elementary Spherical Trigonometry. It is only great circles, then, that are objects of Spherical Trigonometry.

§ 68. *Proposition.* Three straight lines that meet in one point, and do not lie in the same plane, determine a spherical triangle; the angle formed by any two of these lines, at the point of intersection, is the measure of the corresponding side of the spherical triangle; and the inclination of any two of the planes, that pass through each pair of the lines, is the measure of the corresponding angle of the spherical triangle; and the intersections of these planes with the surface of the sphere, describe upon it three arcs of great circles, forming a spherical triangle.

Demonstration. In figure 9, AB, AB'. AB'' being three straight lines, in different planes, and A their common point of intersection; if the centre of a sphere be placed at A, the lines AB, AB', AB'', or their prolongations, will determine three points on the surface of the sphere, as C, C', C'', for we have $AC = AC' = AC''$; and the arcs CC', $C'C''$, CC'', are parts of the circumferences of the great circles passing through the planes CAC', $C'AC''$, and CAC'', respectively, and determined by the lines AC, AC'; AC', AC''; and AC, AC''; a spherical triangle, $CC'C''$, will therefore be determined upon the surface of the sphere, by the intersection of this surface with the sides of the triangular pyramid whose summit is at the centre of the sphere. The inclination of the planes that pass through each pair of these lines, just mentioned, are (Euclid, Book xi. Def. 6) measured by the angles made by the perpendiculars erected in each plane from some point in their common intersection. The arcs CC', $C'C''$, CC'', as well as their respective tangents, are perpendicular to this common intersection, each in its plane respectively; therefore the inclinations of these planes are the measure of the corresponding angles of the spherical triangle.

We therefore have, in any spherical triangle, $CC'C''$, the following results, viz:

arc CC' $=$ angle BAB'
arc $C'C''$ $=$ angle $B'AB'$
arc CC'' $=$ angle BAB''

spheric angle $CC'C'' =$ angle of inclination of BAB', and $B'AB''$;
" $C'CC'' =$ " " $B'AB$, " BAB'';
" $C'C''C =$ " " $B'AB''$, " $B''AB$.

From what has been stated, we learn:

1*st*. That all the parts of a spherical triangle, whether angles or sides, may be expressed by the trigonometric functions, that have constituted the object of the investigations of the First Part of this treatise; for they are all angles, or arcs, the measures of angles.

2*d*. That the principles of Solid Geometry, that are applicable to the sphere and triangular pyramid, are also the first principles of Spherical Trigonometry.

§ 69. Pursuing the application of Solid Geometry, we obtain a series of lemmata, to be used as fundamental principles of Spherical Trigonometry.

Lemma 1. Two great circles of a sphere cannot cut one another, except in one of their diameters; because their line of intersection must pass through the centre of the sphere, which is a point common to them all. The arcs described upon the surface of the sphere, between these intersections, will therefore be equal to two right angles, or 180°.

Lemma 2. If a line be drawn through the centre of the sphere perpendicular to any two of the lines, as AB'' and AB', this line will be a perpendicular (or *normal*) to the plane passing through these lines; and consequently to the great circle of which $C'C''$ is a portion; and all the arcs on the surface of the sphere, intercepted between this perpendicular and the great circle passing through $C'C''$, will be $= 90°$; or represented by a right angle at the centre of the sphere A. (Euclid, B. xi. Prop. 18.)

As this is the case on both sides of the great circle passing through $C'C''$, two points are thus determined by it on the surface of the sphere, that are called the poles, P, p, *figure* 10, of the great circle passing through $C'C''$.

We have thence:

$$PC' = PC'' = PD = pC' = pC'' = pD = \llcorner R = 90^\circ$$

And the same is true of any arc whatever, drawn from P, or p, to the great circle, $DC'C''$.

Lemma 3. Every plane that passes through the line PAp, is perpendicular to this circle; and hence, every great circle of the sphere passing through the poles, Pp, is perpendicular to the circle, $DC'C''$, since these circles are intersections of the planes passing through Pp, and the surface of the sphere; and, conversely, every great circle perpendicular to the circle through $DC'C''$, passes through the poles, P, p. (Euclid, B. xi. Prop. 18.)

Lemma 4. The plane that passes through the tangents, of two great circles, that cut each other in the point P, or p, *figure* 10, is parallel to the great circle of which these points are the poles; for the diameter PAp, is perpendicular to both these planes. (Euclid, B. xi. Prop. 14.)

Lemma 5. The angles which two arcs, such as PC' and PC'', *figure* 10, make at the pole of a great circle, is equal to the arc, $C'C''$, of the great circle intercepted by these arcs; for it is the measure of the angle of inclination of their planes; and the angles at the two poles are equal; since they are measured by the same arc, the same circumstances taking place at both poles. (Euclid, B. xi. Prop. 10.)

Lemma 6. The angle subtended by the poles of two great circles is equal to the angle of inclination of the planes of the two circles of which they are the poles; for these two circles are perpendicular to their axes that pass through the poles. Or, in *figure* 11, the arc Pp is equal to the arc BC, which measures the inclination of these planes, and lies in the same plane with Pp; because AP, Ap, AB, AC, are all perpendicular to the common intersection, Dd, of the two circles, (Euclid, B. xi. Prop. 5) and PAB, and pAC, are both right angles, having the part pAB common; therefore, their complements are equal, or $Pp = BC$.

Lemma 7. If an arc of a great circle fall upon another arc of a great circle, the sum of the two angles, which it makes with this arc, on the two sides, are equal to two right angles; for they are measured by the angles of the tangents to these arcs. And the sum of all the angles made by any number of arcs of great circles, cutting one another at the same point, is always equal to four right angles; and also, the vertical angles made by two arcs, intersecting each other, are always equal; for they are all measured by the tangents to these arcs, and these tangents lie in one plane, being all perpendicular to the radius of the sphere; and thus the Propositions, Euclid, B. i. Prop. 13 and 15, become applicable.

Lemma 8. The perpendicular arc, *Pcd, figure* 10, drawn through one of the angular points of the spherical triangle to the opposite side, is a part of a great circle, passing through this angular point, and the poles of the great circle, of which the opposite side is a portion; for no other plane, than one passing through its poles, can be perpendicular to a great circle.

Lemma 9. Two great circles that cut each other, enclose between them a portion of the spherical surface, that has the same ratio to the surface of the sphere, that the angle of their inclination has to four right angles; for the surface of this section is measured by (or proportional to) the angle of the inclination of the planes of these great circles; and the whole circumference is measured by four right angles; that is to say: as we have the circumference of the great circle $= 2\,r\,\pi$, the whole surface of the sphere is represented, according to these considerations, by

$$S = 2 . r^2 . \pi . 4 . \llcorner R$$

And any section of the surface, measured by the angle a, is

$$s = 2 . r^2 . \pi . a$$

$$\text{whence} \qquad S : s = 4 \llcorner R : a.$$

§ 70. All the propositions of elementary geometry, that determine the data that are necessary to infer the equality and proportionality between the parts or surfaces of plane triangles, are true of spherical triangles; except, that the exterior angle of a spherical triangle is not equal to the sum of the two interior and opposite angles, because the three angles of the spherical triangle do not lie in the same plane; for this reason, the knowledge of two angles of a spherical triangle does not imply the determination of the magnitude of the third, as in plane trigonometry; while, on the other hand, if any three parts of a spherical triangle be given, the remaining three may be determined, even although no one of the parts given be a side. This last follows from the circumstance, that all the quantities concerned in a spherical triangle are of the same nature. All these are evident consequences of the preceding sections, which show that the spheric triangle is the intersection of the triangular pyramid with the surface of the sphere; all that would apply to the plane triangular base of the pyramid must therefore also be true for the spheric base, with the exception that has been stated, which is evidently a consequence of what has been already said.

§ 71. *Theorem.* The sum of all the sides of a spherical triangle is always less than four right angles; and any one side is always less than the sum of the other two sides.

Demonstration. The three planes that intersect each other in the lines *AC*, *AC'*, *AC''*, *figure* 9, form at the point *A*, or the centre of the sphere, a solid angle; the sum of all the plane angles forming a solid angle around a point, is always less than four right angles. (Euclid. B. xi. Prop. 21.) As the three sides of the triangle formed upon the surface of the sphere, by the intersection of these planes, are the measures of the angles at the centre, (section 68,) their sum is also less than four right angles. For the same reason it follows (from Euclid, B. xi. Prop. 20) that the sum of any two of the sides is greater than the third side.

§ 72. *Theorem.* The sum of the three angles of a spherical triangle is always less than six, and more than four right angles.

Demonstration. The sum of the angles forming the three solid angles, at the three angular points of the spherical triangle enclosed by the planes passing through its sides and the plane through the tangents at the sphere, (or the spherical surface, as these angles are the same,) is less than three times four right angles, (or twelve right angles,) for each of them is less than four right angles. (Euc. B. xi. Pr. 21.) But the angles formed by the tangents and the intersections of the planes, are right angles, and their sum is three times two, or six, right angles. The sum of the remaining three angles, formed by the tangents, which are the same with the spherical angles, is therefore less than six right angles.

Or, more briefly and algebraically:

Calling S = sum of the angles forming the 3 solid angles;
" S' = sum of the angles of the triangle, or of the tangents; we have:

$$S = S' + 6 \llcorner R$$

$$S \angle 12 . \llcorner R$$

therefore $$S' + 6 \llcorner R \angle 12 \llcorner R$$

Subtracting $6 \llcorner R$ from both sides:

$$S' \angle 6 \llcorner R$$

$Q : E : D$

If the three angles were equal together to two right angles, the three sides would be in the same plane; the lines of intersection of the planes perpendicular to the tangents would be parallel to each other, and would no longer meet in the centre of the sphere, which is contrary to the supposition; therefore, the sum of the spherical angles must be *more* than four right angles.

§ 73. *Theorem.* If, from the three angular points of a triangle, arcs be drawn on the surface of the sphere, whose distances from the angular points are each = 90°, the intersections of these arcs will form a new triangle, that is called supplementary; that is to say: the sides of this new triangle will be the supplements of those angles of the original triangle, that are opposite to them; and the angles of the new triangle will be the supplements of the sides of the original triangle, which are opposite to them. Moreover, the angular points of the original triangle will be the poles of the sides of the new triangle; and the angular points of this, will be the poles of the sides of the original triangle. They, therefore, are also called polar triangles, in respect to each other.

Construction. In *figure* 12, let *ABC* be the spherical triangle from whose angular points the arcs *DE*, *EF*, *FD*, are drawn, at the distance of 90°; the points *A*, *B*, *C*, will be respectively, the poles of the arcs that cut each other in the points *D*, *E*, *F*; and form the supplementary triangle *DEF*.

Produce the sides of the original triangle, *ABC*, until they intersect the arcs *DE*, *FE*, *FD*, in the points *G*, *H*, *L*, *M*, *J*, *K*.

Demonstration. Because the point *E* is 90° distant from each of the two points *A*, and *B*, this point *E*, is the pole of the circle *LABG*, that passes through the points *A* and *B*; for the same reason, the point *F* is the pole of the arc *KBCM*; and the point *D*, the pole of the arc *JACH*; whence we have:

$$DH + EG = DE + GH = 2 \llcorner R = 180^\circ$$

But, because *GH* is an arc of a circle distant 90° from the point *A*, it is the measure of the spherical angle at *A*, formed by the two arcs *AG*, *AH*; and *DE* is the side of the supplementary triangle that is opposite to the angle *BAC*; for which reasons we have:

$$DE = 180^\circ - GH = 180^\circ - A.$$

In like manner we obtain:

$$EF = 180^\circ - LM = 180^\circ - B.$$

$$\text{and} \quad DF = 180^\circ - JK = 180^\circ - C.$$

Moreover, since the arc $EL = EG = 90^\circ$; and *E* is the pole of the arc $GBAL$, or AB, produced until it intersects the other arcs EF, and ED; the arc GL is the measure of the angle at E, of the supplementary triangle; whence we have:

$$LB + AG = LG + AB = 2 \llcorner R = 180^\circ.$$

$$\text{therefore} \quad LG = 180^\circ - AB.$$

AB, then, is the supplement of LG; or of the angle E, which it measures, or is equal to. For the same reason, we have for all the angles of the supplementary triangle, expressed by the sides of the original triangle:

$$LG = E = 180^\circ - AB.$$
$$HG = D = 180^\circ - AC.$$
$$KM = F = 180^\circ - BC.$$

Therefore, generally, the sides and angles of the supplementary triangle are the supplements of the angles and sides of the original triangle.

§ 74. *Theorem.* The surface of a spherical triangle is proportional to the spherical excess of its three angles above two right angles.

Construction. Let abc, *figure* 13, be a spherical triangle. Produce ac, until the entire circumference of the great circle $aced$, be completed. Produce also ab, and bc, until they cut this great circle, in the points d, and e, and also to their common intersection, in f, on the opposite side of the great circle $cade$, which point f, will be the opposite pole to the point b. (*Lemma* 1.)

Demonstration. By construction, $af = bc$, as they are each of them a supplement of the arc ab; for

$$abe = 2 \llcorner R = baf;$$

for the same reason, the arc $cf = bd$; and the angles at f, and b, are equal; for they are at the opposite poles, and between the same arcs; wherefore the triangles bde, and fac, are equal.

The hemisphere whose base is the great circle *cade*, is, *by lemma* 9, equal or proportional to (calling H = hemisphere)

$$H = 2.r^2.\pi\, 2\llcorner R$$

The section of the sphere between the two semicircles *baf*, *bcf*, is equal, or proportional to:

$$abc + acf = abc + deb = 2\, r^2\, \pi\, b$$

because the angle b is the measure of the inclination of the planes of these two circles.

For the same reason we have:

The spheric section; $dbcad = abc + abd = 2\, \pi\, r^2 . c$

" " $ebace = abc + cbe = 2\, \pi\, r^2 . a$

Subtracting from the sum of all these, twice the value of the triangle *abc*, which is contained in each of them, and of course three times in their sum, the whole hemisphere is represented thus:

$$H = daced = 2.r^2\, \pi \llcorner R$$

$$= 2.r^2\, \pi\, b + 2.r^2\, \pi\, c + 2\, r^2\, \pi\, a - 2\, abc$$

whence

$$2.r^2\, \pi\, 2.\llcorner R = 2.r^2\, \pi\, (b + c + d) - 2\,.abc$$

or

$$r^2\, \pi\, .\, 2\llcorner R = r^2\, \pi\, (b + c + a) - abc$$

And transposing:

$$abc = \pi\, r^2\, (a + b + c) - 2\, \pi\, r^2 \llcorner R$$

$$= \pi\, r^2\, (a + b + c - 2\llcorner R)$$

The surface of the spherical triangle is equal to the product of the square of the radius of the sphere into the excess of the three spherical angles above two right angles; and as we always assume the radius = 1, (until applied to a determinate sphere,) we have:

$$abc = a + b + c - 2\llcorner R$$

CHAPTER II.

Investigation of the Fundamental Formulæ of Spherical Trigonometry; and Solutions of Right Angled Spherical Triangles.

§ 75. *General Problem.* To determine the relations between the sides and angles of a right angled spherical triangle.

Let *DA*, *DB*, *DC*, *figure* 14, be three lines, that are not in the same plane, and which, by their intersection at the point *D*, determine the spherical triangle *ABC*; and let the plane *BDA*, be perpendicular to the plane *ADC*; these, by the preceding chapter, are the elements of a spherical triangle, *ABC*, right angled at A.

Construction. In the line *DB*, take any point, *E*, and from it draw *EG*, perpendicular to *DC*; and from *G*, where *EG* intersects the *DC*, draw, in the plane *ADC*, *GF*, perpendicular to *DC*; join *EF*; the angle *EGF* will be the angle of inclination of the planes *BDC*, and *ADC*, and therefore equal to the spherical angle *BCA*.

Solution. By construction, *CD* is perpendicular to the plane passing through *EGF*; for a like reason, the plane of *DGF*, that passes through this perpendicular, is perpendicular to the plane *EGF*; and by supposition, the plane *BDA*, is perpendicular to the plane *CDA*; therefore the two planes *BDA*, and *EGF*, are perpendicular upon *CDA*, and their common intersection, *EF*, is also perpendicular to *CDA*; and the angles *EFD*, and *EFG*, are right angles; and the four triangles, *DGE*, *DGF*, *DFE*, and *GFE*, are all right angled, at the points *G*, and *F*. (Euc. B. xi. Prop. 4, 18, 19.)

Each pair of these triangles has one side common, and one angle (a right angle) equal in each; wherefore, if two sides, in any one of the triangles, be given, two sides in an other, may be determined; and also: the determination of two of the triangles will afford the means of determining a third.

Determining, then, upon the principles of section 7, the various relations of the parts of these triangles; keeping it constantly in mind: that the *ratios* of the sides of a right angled triangle, and not their absolute values, determine the trigonometric functions; if we adopt, for the sake of brevity, the following notation:

$$EF = c' ; \quad EG = a' ; \quad GF = b' ;$$
$$DE = g ; \quad DF = e ; \quad DG = f ;$$

And for the arcs; $BC = a$; $AB = c$; $AC = b$;

we have as follows:

1. Determining the triangle ***EGF***, by means of the triangles ***DEF***, and ***DEG***:

$$\frac{EF}{ED} = \frac{c'}{g} = \sin BDA = \sin BA = \sin c$$

and $$\frac{EG}{ED} = \frac{a'}{g} = \sin BDC = \sin BC = \sin a$$

And by division, (using only the denominations adopted above:)

$$\frac{c'}{g} : \frac{a'}{g} = \frac{c'}{a'} = \frac{\sin c}{\sin a}$$

In the triangle ***EFG*** we have also (according to section 68:)

$$\frac{EF}{EG} = \frac{c'}{a'} = \sin EGF = \sin C$$

From these two results we obtain the following equation:

$$\frac{c'}{a'} = \frac{\sin c}{\sin a} = \sin C$$

or $$\sin c = \sin C \sin a$$

2. Determining DGF, by means of DEF, and DEG, we have:

$$\frac{DF}{DE} = \frac{e}{g} = \cos BDA = \cos BA = \cos c$$

and

$$\frac{DG}{DE} = \frac{f}{g} = \cos BDC = \cos BC = \cos a$$

Dividing the second by the first:

$$\frac{f}{g} : \frac{e}{g} = \frac{f}{e} = \frac{\cos a}{\cos c}$$

And by the triangle DGF:

$$\frac{DG}{DF} = \frac{f}{e} = \cos ADC = \cos AC = \cos b$$

And by equality from these two results:

$$\frac{f}{e} = \frac{\cos a}{\cos c} = \cos b$$

or

$$\cos a = \cos b \cos c \qquad 2$$

3. Determining DGF, by EGF, and EGD, we obtain:

$$\frac{EG}{DG} = \frac{a'}{f} = \tan BDC = \tan BC = \tan a$$

and

$$\frac{FG}{DG} = \frac{b'}{f} = \tan ADC = \tan AC = \tan b$$

And by division of the second by the first:

$$\frac{b'}{f} : \frac{a'}{f} = \frac{b'}{a'} = \frac{\tan b}{\tan a}$$

By the triangle DGF:

$$\frac{b'}{a'} = \cos EGF = \cos C$$

Hence by equality:

$$\frac{b'}{a'} = \frac{\tan b}{\tan a} = \cos C$$

3 or $$\tan b = \tan a \cos C$$

4. Determining *EGF*, by *DGF*, and *DEF*:

$$\frac{EF}{DF} = \frac{c'}{e} = \tan BDA = \tan BA = \tan c$$

and $$\frac{GF}{DF} = \frac{b'}{e} = \sin ADC = \sin AC = \sin b$$

And by division:

$$\frac{c'}{e} : \frac{b'}{e} = \frac{c'}{b'} = \frac{\tan c}{\sin b}$$

By the triangle *EGF*:

$$\frac{c'}{b'} = \tan EGF = \tan C$$

Whence by equality:

$$\frac{c'}{b'} = \frac{\tan c}{\sin b} = \tan C$$

4 or $$\tan c = \tan C \sin b$$

§ 76. If we substitute for the angle *C*, in such of the preceding formulæ as contain it, the angle *B*, and for the sides of the triangle, those which have the same position in relation to the angle *B*, as the sides named in the preceding formulæ, have in relation to the angle *C*; the formulæ No. 1, 3, and 4, will give also the following:

From No. 1, is obtained; $\sin a \sin B = \sin b$ 5

3, $\cos B \tan a = \tan c$ 6

4, $\sin c \tan B = \tan b$ 7

The formula No. 2 cannot be changed, because it always expresses the relation of equality that exists between the cosine of the hypothenuse, and the product of the cosines of the sides that contain the right angle.

The comparison of the formulæ already obtained in this chapter, gives, by means of the equalities that are found in them, the following formulæ, which complete all the cases of right angled spherical trigonometry.

Comparing No. 4 with No. 6:

$$\tan c = \sin b \tan C = \cos B \tan a$$

Substituting from No. 5: $\sin b = \sin a \sin B$

$$\sin a \sin B \tan C = \cos B \tan a$$

Dividing by: $\sin B \tan a$:

$$\cos a \tan C = \cot B \qquad 8$$

By substituting, in the same equation, from No. 3, $\tan a = \dfrac{\tan b}{\cos C}$, the equation becomes:

$$\sin b \tan C = \frac{\cos B \tan b}{\cos C}$$

Multiplying by $\cos C \cot b$:

$$\cos b \sin C = \cos B \qquad 9$$

§ 77 The nine formulæ of the two preceding sections may be reduced to six, by remarking: that No. 5, 6, and 7, are mere repetitions of No. 1, 3, and 4; for they differ only in having relation to the other oblique angle of the triangle. They give all the cases of right angled spherical trigonometry. It may be of use to repeat these principal formulæ, and to transform them into other similar ones, applicable to a triangle that has one side = 90°, by means of the supplementary triangle, whose properties have been explained in section 73.

b We have then:

1 $\sin c = \sin a \sin C$

2 $\cos a = \cos b \cos c$

3 $\tan b = \tan a \cos C$

4 $\tan c = \sin b \tan C$

5 $\cot B = \cos a \tan C$

6 $\cos B = \cos b \sin C$

Transforming these equations to a triangle, one of whose sides is = 90°, by means of the supplementary triangle, and using the same characteristic characters, merely accentuated, we have:

7 $\sin C' = \sin A' \sin c'$

8 $\cos A' = \cos B' \cos C'$

9 $\tan B' = \tan A' \cos c'$

10 $\tan C' = \sin B' \tan c'$

11 $\cot b' = \cos A' \tan c'$

12 $\cos b' = \cos B' \sin c'$

The signs of the trigonometric functions have no influence in these mutations.

CHAPTER III.

Investigation of the General Formulæ of Oblique Angled Spherical Trigonometry.

§ 78. *General Problem.* To find the relations between the parts of an oblique angled spherical triangle.

Construction. Let ABC, *figures* 15 and 16, be a spherical triangle, that has either three acute angles, or the angle at A, obtuse.

From C, draw the arc CD, perpendicular to BA; it will in the first case, be opposite to the two interior angles A, and B; and in the second, to the interior angle at B, and the exterior at A, or to the supplement of the interior angle A; the functions of which exterior angle are the same with those of the interior angle A, of the triangle.

Solution. The two right angled triangles, CBD, and CAD, formed by the perpendicular CD, have this perpendicular common to both. If, then, we express any one of the functions of this side by the functions of the other parts of each of the triangles, we shall obtain equations between the functions of these parts, which may be transformed into proportions, and which will furnish all the solutions of the oblique angled spherical triangle ABC, by means of its parts.

Performing this process for the sine, the cosine, and the tangent of CD, we obtain the following table; which follows as the direct consequence of the formulæ b, No. 1 to 6, of section 77. We have therefore :

	By the triangle BCD :	By the triangle ACD :	c
$\sin CD$	$= \sin a \,.\, \sin B$	$= \sin b \,.\, \sin A$	1
	$= \tan BD : \tan BCD$	$= \tan AD : \tan ACD$	2
$\cos CD$	$= \cos a : \cos BD$	$= \cos b : \cos AD$	3
	$= \cos B : \sin BCD$	$= \cos A : \sin ACD$	4
$\tan CD$	$= \tan a \,.\, \cos BCD$	$= \tan b \,.\, \cos ACD$	5
	$= \sin BD \,.\, \tan B$	$= \sin AD \,.\, \tan A$	

As the first formula contains no other terms than the parts of an oblique angled triangle, it is evident that its result is general; for an analogous result would be obtained, if the perpendicular were let fall from one of the other angles of the triangle, upon its opposite side; we therefore have likewise:

7 $$\sin c \sin B = \sin C \sin b$$

8 $$\sin C \sin a = \sin A \sin c$$

These formulæ, (1, 7, and 8,) transformed into proportions, give:

9 $$\sin a : \sin b = \sin A : \sin B$$

10 $$\sin c : \sin b = \sin C : \sin B$$

11 $$\sin a : \sin c = \sin A : \sin C$$

By which it appears that we have, in Spherical Trigonometry, a general principle analogous to that found in Plane Trigonometry; viz: that *the sines of the sides are proportional to the sines of the opposite angles.*

The formulæ 5 and 6, transformed into proportions, give the following results, viz:

12 From No. 5; $\tan a : \tan b = \cos ACD : \cos BCD$

13 " 6; $\tan A : \tan B = \sin BD : \sin AD$

Corollary. The sines of the perpendiculars let fall from the different angular points upon the opposite sides in a spherical triangle, are to one another inversely as the sines of those sides.

Demonstration. We had by c, No. 1, (*figure* 17:)

$$\sin CD = \sin a \sin B$$

Drawing from the point A, an arc perpendicular to BC, we have from the same principles:

$$\sin AD' = \sin c \sin B$$

therefore

14 $$\sin CD : \sin AD' = \sin a : \sin c$$

§ 79. Considering the formulæ c, No. 2, 3, 4, 9, 12, and 13, as proportions; and combining them by addition and subtraction, as allowed by the principles of proportion; we deduce a series of formulæ, useful in the transformations of trigonometric formulæ, to adapt them for calculation. They also form, in conjunction with the original proportions from which they are deduced, a series of solutions of a spherical triangle, by means of its parts, formed by a perpendicular drawn from one of its angles on the opposite side. When the angle is obtuse, it is known by the algebraic sign of the trigonometric functions.

By this operation we obtain the following results in succession, in which we adopt for the statement of proportions, the mode of writing them as equations of fractions; (as more convenient;) and, to render the process more easy, we assume the following notations, in addition to those already mentioned; viz:

$$AD = c_{\prime}\ ; \qquad BD = c_{\prime\prime}\ ;$$
$$ACD = C_{\prime}\ ; \qquad BCD = C_{\prime\prime}$$

whence

$$AD + BD = c_{\prime} + c_{\prime\prime} = c$$
$$ACD + BCD = C_{\prime} + C_{\prime\prime} = C$$

We have by c, No. 9:

$$\frac{\sin a}{\sin b} = \frac{\sin A}{\sin B}$$

Adding and subtracting this proportion by the well known method, we have:

$$\frac{\sin a + \sin b}{\sin a \backsim \sin b} = \frac{\sin A + \sin B}{\sin A \backsim \sin B}$$

And substituting from series M, No. 7 and 10:

$$\frac{\sin \tfrac{1}{2}(a + b) \cos \tfrac{1}{2}(a \backsim b)}{\cos \tfrac{1}{2}(a + b) \sin \tfrac{1}{2}(a \backsim b)} = \frac{\sin \tfrac{1}{2}(A + B) \cos \tfrac{1}{2}(A \backsim B)}{\cos \tfrac{1}{2}(A + B) \sin \tfrac{1}{2}(A \backsim B)}$$

d or

1
$$\frac{\tan \frac{1}{2}(a+b)}{\tan \frac{1}{2}(a \backsim b)} = \frac{\tan \frac{1}{2}(A+B)}{\tan \frac{1}{2}(A \backsim B)}$$

From c, No. 2, is obtained by the same means:

$$\frac{\tan c_{,,} + \tan c_{,}}{\tan c_{,,} \backsim \tan c_{,}} = \frac{\tan C_{,,} + \tan C_{,}}{\tan C_{,,} \backsim \tan C_{,}}$$

Substituting from G, No. 1:

$$\frac{\sin (c_{,,} + c_{,})}{\sin (c_{,,} \backsim c_{,})} = \frac{\sin (C_{,,} + C_{,})}{\sin (C_{,,} \backsim C_{,})}$$

or

2
$$\frac{\sin c}{\sin (c_{,,} \backsim c_{,})} = \frac{\sin C}{\sin (C_{,,} \backsim C_{,})}$$

From c, No. 3, is obtained by this method:

$$\frac{\cos a + \cos b}{\cos a \backsim \cos b} = \frac{\cos c_{,,} + \cos c_{,}}{\cos c_{,,} \backsim \cos c_{,,}}$$

Substituting from series M, No. 8 and 11:

$$\frac{\cos \frac{1}{2}(a+b) \cos \frac{1}{2}(a \backsim b)}{\sin \frac{1}{2}(a+b) \sin \frac{1}{2}(a \backsim b)} = \frac{\cos \frac{1}{2} c \cos \frac{1}{2}(c_{,,} \backsim c_{,})}{\sin \frac{1}{2} c \sin \frac{1}{2}(c_{,,} \backsim c_{,})}$$

or

3
$$\frac{\cot \frac{1}{2}(a+b)}{\tan \frac{1}{2}(a \backsim b)} = \frac{\cot \frac{1}{2} c}{\tan \frac{1}{2}(c_{,,} \backsim c_{,})}$$

From c, No. 4, is obtained by this method:

$$\frac{\cos B + \cos A}{\cos B \backsim \cos A} = \frac{\sin C_{,,} + \sin C_{,}}{\sin C_{,,} \backsim \sin C_{,}}$$

And substituting from M, No. 7, 8, 10, and 11:

$$\frac{\cos \frac{1}{2}(A+B) \cos \frac{1}{2}(A \backsim B)}{\sin \frac{1}{2}(A+B) \sin \frac{1}{2}(A \backsim B)} = \frac{\sin \frac{1}{2} C \cos \frac{1}{2}(C_{,,} \backsim C_{,})}{\cos \frac{1}{2} C \sin \frac{1}{2}(C_{,,} \backsim C_{,})}$$

or

$$\frac{\cot \frac{1}{2}(A + B)}{\tan \frac{1}{2}(A \backsim B)} = \frac{\tan \frac{1}{2} C}{\tan \frac{1}{2}\ C_{,,} \backsim C_{,})} \qquad 4$$

From c, No. 12, is obtained:

$$\frac{\tan a + \tan b}{\tan a \backsim \tan b} = \frac{\cos C_{,} + \cos C_{,,}}{\cos C_{,} \backsim \cos C_{,,}}$$

Substituting from G, No. 1, and M, No. 8 and 11:

$$\frac{\sin (a + b)}{\sin (a \backsim b)} = \frac{\cot \frac{1}{2} C}{\tan \frac{1}{2}(C_{,,} \backsim C_{,})} \qquad 5$$

From c, No. 13, is obtained in a similar manner:

$$\frac{\sin c_{,,} + \sin c'}{\sin c_{,,} \backsim \sin c_{,}} = \frac{\tan A + \tan B}{\tan A \backsim \tan B}$$

Substituting from M, No. 7 and 10, and G, No. 1:

$$\frac{\sin \frac{1}{2} c \cos \frac{1}{2}(c_{,,} \backsim c_{,})}{\cos \frac{1}{2} c \sin \frac{1}{2}(c_{,} \backsim c_{,,})} = \frac{\sin (A + B)}{\sin (A \backsim B)}$$

or

[*] $$\frac{\tan \frac{1}{2} c}{\tan \frac{1}{2}(c_{,,} \backsim c_{,})} = \frac{\sin (A + B)}{\sin (A \backsim B)} \qquad 6$$

§ 80. The section 78 has already given one of the general principles for the solution of Spheric Trigonometry, applicable to the cases, where two sides and one angle, or two angles and one side are given; so that one of the given angles and sides are opposite to each other, and the part sought opposite to the remaining part given.

To obtain other such formulæ, in which the three parts given are two sides and the included angle, and the part sought one of the other angles; or when two angles and the included side are given, and one of the other sides sought; we have not here given, as in Plane Trigonometry, the sum of the remaining angles. A formula that is applicable to this case, and to those derived from it, may be obtained in the following manner:

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[*] In these formulæ and their combinations lie all those called Napier's analogies, which it is not here necessary to treat in full.

We have from a, No. 3. by transforming the denominations into those here employed:

$$\tan c_{\prime\prime} = \tan a \cos B$$

By F, No. 1:

$$\sin c_{\prime} = \sin (c \backsim c_{\prime\prime}) = \sin c \cos c_{\prime\prime} \backsim \cos c \sin c_{\prime\prime}$$

Whence

$$\frac{\sin c_{\prime}}{\sin c_{\prime\prime}} = \frac{\sin c}{\tan c_{\prime\prime}} \backsim \cos c$$

By c, No. 13:

$$\frac{\sin c_{\prime}}{\sin c_{\prime\prime}} = \frac{\tan B}{\tan A}$$

Therefore, by equality:

$$\frac{\tan B}{\tan A} = \frac{\sin c}{\tan c_{\prime\prime}} \backsim \cos c$$

Substituting for tangent $c_{\prime\prime}$ the first of the above formulæ:

$$\frac{\tan B}{\tan A} = \frac{\sin c}{\tan a \cos B} \backsim \cos c$$

Reducing to a common denominator gives, after compensating this common denominator on both sides:

$$\tan a \sin B = \sin c \tan A \backsim \cos c \tan a \cos B \tan A$$

e Multiplying by: cotangent a cotangent A:

1 $$\cot A \sin B = \sin c \cot a \backsim \cos c \cos B.$$

As this formula contains the functions of two sides and two angles, of which one is opposite and an other included, alternately; if any three of these parts be given, the fourth may be found; it is, therefore, general in all cases, as has been shown to be true of formula c, No. 1, for those cases where the sides and angles are mutually opposite.

§ 81. In order to complete all the possible *modes* of combining the six parts of a spherical triangle, there only remains to be investigated, an analytic expression that shall contain three parts of the same kind, and one of a different kind; that is, three sides and one angle, or three angles and one side. A formula of this kind may be obtained by means of a process similar to that of section 80, with merely an appropriate variation in the parts substituted.

By a No. 3, we have:

$$\tan c_{,,} = \tan b \cos A$$

By F, No. 2:

$$\cos c_{,} = \cos (c \backsim c_{,,}) = \cos c \cos c_{,,} + \sin c \sin c_{,,}$$

Whence

$$\frac{\cos c_{,}}{\cos c_{,,}} = \cos c + \sin c \tan c_{,,}$$

By c, No. 3, we have also, (substituting the notation here used:)

$$\frac{\cos c_{,}}{\cos c_{,,}} = \frac{\cos a}{\cos b}$$

Thence by equality:

$$\frac{\cos a}{\cos b} = \cos c + \sin c \tan c_{,,}$$

And substituting for tangent $c_{,,}$, the first formula above:

$$\frac{\cos a}{\cos b} = \cos c + \sin c \tan b \cos A$$

Or finally, (multiplying by cosine b:)

2 $$\cos a = \cos c \cos b + \sin c \sin b \cos A$$

§ 82. The general formulæ c, No. 1, and e, No. 1, are of an identical nature; the sides and angles appearing in them in an even number, or symmetric. It remains for us to show, that the formula e, No. 2, just preceding, has the same property of generality, as relates to its application to sides or angles, when proper attention is paid to the consequence of the change in the assumption; this may be most easily shown by a reference to the supplementary triangle.

Transforming, therefore, e, No. 2, in conformity with the principles of the supplementary triangle, we shall have all the sides changed into angles, and the angle will become a side; the formula will thus be adapted to three angles and one side; but it must be observed: that the cosines of the supplementary angle are negative; and that the algebraic signs of the different terms undergo the change consequent to this change of sign. If the transformation be performed with regard to this circumstance, and the single term on the left hand rendered positive, as is always usual for the part sought in an equation; we shall obtain the result:

3 $$\cos A = \sin C \sin B \cos a - \cos C \cos B$$

§ 83. As the three general formulæ, series c, No. 1, and series e, No. 1 and 2, include the analytical expressions of all the cases of oblique angled spherical trigonometry; it will be proper, on account of the frequent use we shall make of them, in deducing from them others, better adapted for calculation in each individual case, to collect them here, so as to be seen at a single glance. It is evident from the above: that, taking them in the full extent of their import, they contain, in the shape of no more than three formulæ, all the solutions of oblique angled spherical trigonometry; that they, therefore, are the fundamental elements of all future investigations of the individual cases.

These three formulæ are as follows :

$$\sin a \sin B = \sin b \sin A \tag{1}$$

$$\cot A \sin B = \sin c \cot a \backsim \cos c \cos B \tag{2}$$

$$\cos a = \sin c \sin b \cos A + \cos c \cos b \tag{3}$$

§ 84. We shall not, therefore, here make any mutations of these formulæ, to adapt them to individual cases, which are, of course, contained in them analytically; but keep them in the full generality of their value and import, all their mutations according to the data of a given case being of course supposed; and proceed in the next chapter to give, for each particular case, a variety of formulæ, adapted to logarithmic calculation, presenting a proper choice, according to the nature of the data of any individual calculation.

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CHAPTER IV.

Deduction of Formulæ adapted to the Logarithmic Calculation of all the cases of Oblique Angled Spherical Trigonometry.

§ 85. This part will here be treated of by means of a complete series of Problems, for each different supposition of parts given and parts sought; applying the appropriate reductions, and using, in case of need, the auxiliary angles; with the modifications of the formulæ whence they are derived; and having reference, for convenience of notation, to a spherical triangle *ABC*, *figure* 15 *or* 16, whose sides *a*, *b*, *c*, are respectively opposite to the angles *A*, *B*, *C*.

§ 86. *Problem* 1. Given, two sides and the angle opposite to one of them, to find the angle opposite to the other side.

b, *c*, and the angle *B*, being given, to find the angle *C*.

By c, No, 7, we have given :

$$\sin B \sin c = \sin C \sin b$$

Therefore

g $$\sin C = \frac{\sin B \sin c}{\sin b}$$

This formula is adapted to logarithmic calculation, without change, as is evident from inspection.

§ 87. *Problem* 2. Given, two angles and a side opposite to one of them, to find the side opposite to the other.

c, B, and C, being given, to find b.

By the same formula we have:

$$\sin B . \sin c = \sin C \sin b$$

Therefore

h $$\sin b = \frac{\sin B \sin c}{\sin C}$$

§ 88. *Problem* 3. Given, the three sides, to find one of the angles.

a, b, c, being given, to find A.

By f, No. 3, we have:

$$\cos a = \sin b \sin c \cos A + \cos b \cos c$$

By transposition and division:

$$\cos A = \frac{\cos a - \cos c \cos b}{\sin b \sin c}$$

1*st Transformation.* To transform this formula for the purpose of adapting it to logarithmic calculation; we have by Q, No. 3: $\cos A = 1 - 2 \sin^2 \frac{1}{2} A$

By which:

$$1 - 2 \sin^2 \tfrac{1}{2} A = \frac{\cos a - \cos c \cos b}{\sin c \sin b}$$

Transposing and reducing to a common denominator, with the consequent change of signs:

$$2 \sin^2 \tfrac{1}{2} A = \frac{\sin c \sin b + \cos c \cos b - \cos a}{\sin c \sin b}$$

And by the two first terms of the numerator:

$$2 \sin^2 \tfrac{1}{2} A = \frac{\cos (c \backsim b) - \cos a}{\sin c \sin b}$$

Substituting from M, No. 11, and dividing both sides of the equation by 2:

$$\sin^2 \tfrac{1}{2} A = \frac{\sin \tfrac{1}{2} \big(a + (c \backsim b)\big) \sin \tfrac{1}{2} \big(a - (c \backsim b)\big)}{\sin c \sin b}$$

Calling $p = \dfrac{a + b + c,}{2}$, as in section 57:

$$\sin^2 \tfrac{1}{2} A = \frac{\sin (p - c) \sin (p - b)}{\sin c \sin b}$$

Whence:

$$\sin \tfrac{1}{2} A = \left(\frac{\sin (p - c) \sin (p - b)}{\sin c \sin b}\right)^{\frac{1}{2}} \qquad 1$$

2d Transformation. We have also by Q, No 4:

$$\cos A = 2 \cos^2 \tfrac{1}{2} A - 1$$

Substituting this, transposing the 1, and reducing to a common denominator, we have by the same process as above:

$$2 \cos^2 \tfrac{1}{2} A = \frac{\cos a - (\cos c \cos b - \sin b \sin c)}{\sin b \sin c}$$

$$= \frac{\cos a - \cos (b + c)}{\sin b \sin c}$$

$$2\cos^2 \tfrac{1}{2} A = \frac{2\sin\frac{1}{2}(a+b+c)\sin\frac{1}{2}(b+c-a)}{\sin b \sin c}$$

2 $$\cos\tfrac{1}{2} A = \left(\frac{\sin p \sin(p-a)}{\sin b \sin c}\right)^{\frac{1}{2}}$$

These formulæ are affected in the same way as the corresponding formulæ of Plane Trigonometry, Y, No. 14 and 15.

3*d Transformation.* We also derive from them, in the way that was there described, the following formula, which is in all cases the most exact in its results.

3 $$\tan\tfrac{1}{2} A = \frac{\sin\frac{1}{2} A}{\cos\frac{1}{2} A} = \left(\frac{\sin(p-c)\sin(p-b)}{\sin p \sin(p-a)}\right)^{\frac{1}{2}}$$

§ 89. *Problem* 4. Given, the three angles, to find one of the sides.

A, B, C, being given, to find a.

By formula e, No. 3:

$$\cos a = \frac{\cos B \cos C + \cos A}{\sin B \sin C}$$

1*st Transformation.* By Q, No. 3: $\cos a = 1 - 2\sin^2 \frac{1}{2} a$ Substituting this value, transposing, and reducing to a common denominator:

$$2\sin^2\tfrac{1}{2} a = \frac{\sin B \sin C - \cos B \cos C - \cos A}{\sin B \sin C}$$

$$= \frac{-\cos(B+C) - \cos A}{\sin B \sin C};$$

By a substitution analogous to M, No. 8, and dividing both sides by 2:

$$\sin^2\tfrac{1}{2} a = -\frac{\cos\frac{1}{2}(B+C+A)\cos\frac{1}{2}(B+C-A)}{\sin B \sin C}$$

Assuming $P = \dfrac{A + B + C}{2}$, and extracting the root:

$$\sin \tfrac{1}{2} A = \left(-\frac{\cos P . \cos (P - A)}{\sin B \sin C}\right)^{\frac{1}{2}}$$

We get rid of the sign —, from the consideration: that P is necessarily an obtuse angle, since the sum of the three angles of a spherical triangle is always greater than two right angles, (by section 72;) hence its half is greater than one right angle, the cosine of which being negative, renders the result positive; we have, therefore, finally:

$$\sin \tfrac{1}{2} a = \left(\frac{\cos P \cos (P - A)}{\sin B \sin C}\right)^{\frac{1}{2}} \qquad 1$$

2d Transformation. By substituting in the above formula $\cos a = 2 \cos^2 \tfrac{1}{2} a - 1$, we have:

$$2 \cos^2 \tfrac{1}{2} a = \frac{\sin B \sin C + \cos B \cos C + \cos A}{\sin B \sin C}$$

$$= \frac{\cos C + \cos (B \backsim C)}{\sin B \sin C}$$

By a substitution analogous to M, No. 8, it becomes:

$$2 \cos^2 \tfrac{1}{2} a = \frac{2 \cos \tfrac{1}{2} \left(A + (B \backsim C)\right) \cos \tfrac{1}{2} \left(A \backsim (B \backsim C)\right)}{\sin B \sin C}$$

$$\cos \tfrac{1}{2} a = \left(\frac{\cos \tfrac{1}{2} (A + B - C) \cos \tfrac{1}{2} (A + C - B)}{\sin B \sin C}\right)^{\frac{1}{2}}$$

$$\cos \tfrac{1}{2} a = \left(\frac{\cos (P - C) \cos (P - B)}{\sin B \sin C}\right)^{\frac{1}{2}} \qquad 2$$

3d Transformation. We have also in this case as in the preceding, from the division of the two formulæ:

3 $$\tan \tfrac{1}{2} a = \left(\frac{\cos P \cos (P - A)}{\cos (P - C) \cos (P - B)}\right)^{\frac{1}{2}}$$

§ 90. *Problem* 5. Given, two sides and the included angle, to find the third side.

b, *c*, and *A*, being given, to find *a*.

We have found by section 88, problem 3:

$$2 \sin^2 \tfrac{1}{2} A = \frac{\cos (c \backsim b) - \cos a}{\sin b \sin c}$$

Or

$$2 \sin b \sin c \sin^2 \tfrac{1}{2} A = \cos (c \backsim b) - \cos a$$

1st Transformation. Substituting for the two cosines on the right, their values from the analogy of Q, No. 3, we have:

$$2 \sin b \sin c \sin^2 \tfrac{1}{2} A = 2 \sin^2 \tfrac{1}{2} a - 2 \sin^2 \tfrac{1}{2} (c \backsim b)$$

Whence

$$\sin^2 \tfrac{1}{2} a = \sin^2 \tfrac{1}{2} (c \backsim b) + \sin b \sin c \sin^2 \tfrac{1}{2} A$$

Making the first term on the right a common factor for both terms, and extracting the square root:

1

1 $$\sin \tfrac{1}{2} a = \sin \tfrac{1}{2} (c \backsim b) \left(1 + \frac{\sin b \sin c \sin^2 \tfrac{1}{2} A}{\sin^2 \tfrac{1}{2} (c \backsim b)}\right)^{\frac{1}{2}}$$

Assuming

2 $$\tan Z = \frac{\sin \tfrac{1}{2} A}{\sin \tfrac{1}{2} (c \backsim b)} (\sin b \sin c)^{\frac{1}{2}}$$

We have for the part under the radical:

$$(1 + \tan^2 Z)^{\frac{1}{2}} = \sec Z = \frac{1}{\cos Z}$$

And the formula becomes:

$$\sin \tfrac{1}{2} a = \frac{\sin \tfrac{1}{2} (c \backsim b)}{\cos Z} \tag{3}$$

2d Transformation. It is evident that it may also be reduced by a substitution analogous to Q, No. 4.

$$2 \sin b \sin c \sin^2 \tfrac{1}{2} A = 2 \cos^2 \tfrac{1}{2} (c \backsim b) - 2 \cos^2 \tfrac{1}{2} a$$

Whence may be deduced by the same process:

$$\cos \tfrac{1}{2} a = \cos \tfrac{1}{2} (c \backsim b) \left(1 - \frac{\sin b \sin c \sin^2 \tfrac{1}{2} A}{\cos^2 \tfrac{1}{2} (c \backsim b)}\right)^{\frac{1}{2}} \tag{4}$$

Assuming

$$\sin Z' = \frac{\sin \tfrac{1}{2} A}{\cos \tfrac{1}{2} (c \backsim b)} (\sin b \sin c)^{\frac{1}{2}} \tag{5}$$

The formula becomes:

$$\cos \tfrac{1}{2} a = \cos \tfrac{1}{2} (c \backsim b) \cos Z' \tag{6}$$

3d Transformation. Taking from problem 3, the formula for the value of $\cos^2 \tfrac{1}{2} A$, instead of the value of $\sin^2 \tfrac{1}{2} A$, we obtain:

$$2 \sin b \sin c \cos^2 \tfrac{1}{2} A = \cos a - \cos (b + c)$$

We have, consequently, the means to transform the equation again in two ways, by means of the two values of cosine a, and cosine $(b + c)$.

In the first place, by a substitution analogous to Q, No. 3, and the transformation made in No. 1:

$$\sin \tfrac{1}{2} a = \sin \tfrac{1}{2} (c + b) \left(1 - \frac{\sin b \sin c \cos^2 \tfrac{1}{2} A}{\sin^2 \tfrac{1}{2} (c + b)}\right)^{\frac{1}{2}} \tag{7}$$

In which, by assuming

$$\sin Z'' = \frac{\cos \tfrac{1}{2} A}{\sin \tfrac{1}{2} (c + b)} (\sin b \sin c)^{\frac{1}{2}} \tag{8}$$

The formula becomes :

9 $$\sin \tfrac{1}{2} a = \sin \tfrac{1}{2} (c + b) \cos Z''$$

4th Transformation. By a substitution analogous to that in the 2d transformation, we also obtain here :

$$2 \sin b \sin c \cos^2 \tfrac{1}{2} A = 2 \cos^2 \tfrac{1}{2} a - 2 \cos^2 \tfrac{1}{2} (c + b)$$

Whence in like manner :

$$\cos^2 \tfrac{1}{2} a = \cos^2 \tfrac{1}{2} (c + b) \left(1 + \frac{\sin b \sin c \cos^2 \tfrac{1}{2} A}{\cos^2 \tfrac{1}{2} (c + b)}\right)$$

10 $$\cos \tfrac{1}{2} a = \cos \tfrac{1}{2} (c + b) \left(1 + \frac{\sin b \sin c \cos^2 \tfrac{1}{2} A}{\cos^2 \tfrac{1}{2} (c + b)}\right)^{\frac{1}{2}}$$

Assuming again

11 $$\tan Z''' = \frac{\cos \tfrac{1}{2} A}{\cos \tfrac{1}{2} (c + b)} (\sin b \sin c)^{\frac{1}{2}}$$

the formula becomes :

12 $$\cos \tfrac{1}{2} a = \frac{\cos \tfrac{1}{2} (c + b)}{\cos Z'''}$$

The four preceding formulæ evidently furnish the means of forming two others for the tangents, as h, No. 3, and i, No. 3, in problems 3, and 4, in the following manner.

5th Transformation. Dividing No. 3, by No. 6 :

13 $$\frac{\sin \tfrac{1}{2} a}{\cos \tfrac{1}{2} a} = \tan \tfrac{1}{2} a = \frac{\sin \tfrac{1}{2} (c \backsim b)}{\cos \tfrac{1}{2} (c \backsim b) \cos Z \cos Z'} = \frac{\tan \tfrac{1}{2} (c \backsim b)}{\cos Z \cos Z}$$

In which we have, as before, the auxiliary angles determined by No. 2 and 5.

6th Transformation. Dividing No. 9, above, by No. 12 :

$$\frac{\sin \tfrac{1}{2} a}{\cos \tfrac{1}{2} a} = \tan \tfrac{1}{2} a = \frac{\sin \tfrac{1}{2} (c + b) \cos Z'' \cos Z'''}{\cos \tfrac{1}{2} (c + b)}$$

14 $$= \tan \tfrac{1}{2} (c + b) \cos Z'' \cos Z'''$$

The values of cosine Z'', and cosine Z''', being determined by No. 8 and 11.

As these two formulæ have the same factors in the auxiliary arcs as the four preceding ones, and represent no more than two different functions, of which the same function is used finally, the calculation of them is not attended with much more labour; and this is fully compensated by their greater exactness and applicability to every case. The nature of the data must in this case, as in every other, determine the choice of the formula; if, for instance, $(c \backsim b)$ were small, the formulæ which employ its cosine would not afford exact results; it is, in general, better to employ those using $(c + b)$; and those giving the tangent, will in all cases be preferable. It will be readily perceived, that it is a matter of indifference whether we take the sine, or cosine, for the auxiliary arc, as well as, whether the tangent or cotangent is used, in the respective cases where their use occurs, provided the proper corresponding functions are also used in the final formula.

7th. Transformation. We may also transform the original formula f, No. 3, of the preceding chapter. This gives a function of the whole angle, as follows; viz:

$$\cos a = \cos c \cos b + \sin b \sin c \cos A$$

by substituting for sine b, or sine c, the value of one or the other, in conformity with the principles whence we deduce series B.

In this manner, writing instead of sine b, its equal, tangent b cosine b, we have:

$$\cos a = \cos b \cos c + \tan b \cos b \sin c \cos A$$

And assuming: $\tan y = \cos A \tan b$, we obtain: 15

$$\cos a = \cos b \cos c + \cos b \sin c \tan y$$

Multiplying the two terms on the right hand by cosine y, we have:

$$\cos a = \frac{\cos b \cos c \cos y + \cos b \sin c \sin y}{\cos y}$$

And, as $\dfrac{\cos b}{\cos y}$ is a common factor:

$$\cos a = \frac{\cos b}{\cos y} (\cos c \cos y + \sin c \sin y)$$

16 $$= \frac{\cos b \,.\, \cos (c \backsim y)}{\cos y}$$

8th *Transformation.* Making, as before,

17 $$\cot y' = \cos A \tan b$$

we finally obtain:

18 $$\cos a = \frac{\cos b \sin (c + y')}{\sin y'}$$

It will be seen that these two formulæ are identical, for the passage from the tangent to the cosine is evidently the same as that from the cotangent to the sine. In the first case, we have the cosine of the difference between the auxiliary angle and the other side; and in the second case, the sine of the sum of the two angles; which produce again the identical trigonometric function.

§ 91. *Problem* 6. Given, two angles and the included side, to find the third angle.

B, *C*, and *a*, being given, to find *A*.

This case is exactly analogous to the preceding, as might, in fact, have been anticipated from the properties of the supplementary triangle; it, notwithstanding, requires a separate investigation, in consequence of the diversity that occurs in the algebraic signs, and in the combination of the simple arcs and their values; which occasions a change of sine into cosine. We therefore give these successive changes.

In preparing the formula k, No. 1, problem 4, we obtained:

$$2 \sin^2 \tfrac{1}{2} a = \frac{-\cos (B + C) - \cos A}{\sin B \sin C}$$

Whence

$$2 \sin B \sin C \sin^2 \tfrac{1}{2} a = -\cos (B + C) - \cos A$$

1*st Transformation.* Introducing the sines and cosines of the half angles instead of the cosines of the whole angles, by substitutions analogous to Q, No. 3 and 4:

$$2 \sin^2 \tfrac{1}{2} A = 2 \cos^2 \tfrac{1}{2} (B + C) + 2 \sin B \sin C \sin^2 \tfrac{1}{2} a$$

Dividing the whole by 2, making $\cos^2 \frac{1}{2} (B + C)$ a common factor, and extracting the root, the formula becomes:

m

$$\sin \tfrac{1}{2} A = \cos \tfrac{1}{2} (B + C) \left(1 + \frac{\sin B \sin C \sin^2 \frac{1}{2} a}{\cos^2 \frac{1}{2} (B + C)}\right)^{\frac{1}{2}} \qquad 1$$

Assuming again

$$\tan Z = \frac{\sin \frac{1}{2} a \, (\sin B \sin C)^{\frac{1}{2}}}{\cos \frac{1}{2} (B + C)} \qquad 2$$

The formula becomes:

$$\sin \tfrac{1}{2} A = \frac{\cos \frac{1}{2} (B + C)}{\cos Z} \qquad 3$$

2*d Transformation.* Substituting sine, and cosine of A, and $(B + C)$, in a manner the inverse of that employed above, we have, as may easily be seen:

$$\cos \tfrac{1}{2} A = \sin \tfrac{1}{2} (B + C) \left(1 - \frac{\sin B \sin C \sin^2 \frac{1}{2} a}{\sin^2 \frac{1}{2} (B + C)}\right)^{\frac{1}{2}} \qquad 4$$

Assuming

$$\sin Z' = \frac{\sin \frac{1}{2} a \, (\sin B \sin C)^{\frac{1}{2}}}{\sin \frac{1}{2} (B + C)} \qquad 5$$

The formula becomes:

$$\cos \tfrac{1}{2} A = \sin \tfrac{1}{2} (B + C) \cos Z' \qquad 6$$

3d Transformation. Taking from the 4th problem the preparation for the second transformation:

$$2 \cos^2 \tfrac{1}{2} a \sin B \sin C = \cos A + \cos (B \backsim C)$$

And substituting the functions of the half angles, exactly as in the preceding transformation, we have:

$$\sin^2 \tfrac{1}{2} A = \cos^2 \tfrac{1}{2} (B \backsim C) \left(1 - \frac{\sin B \sin C \cos^2 \tfrac{1}{2} a}{\cos^2 \tfrac{1}{2} (B \backsim C)}\right)$$

7 $$\sin \tfrac{1}{2} A = \cos \tfrac{1}{2} (B \backsim C) \left(1 - \frac{\sin B \sin C \cos^2 \tfrac{1}{2} a}{\cos^2 \tfrac{1}{2} (B \backsim C)}\right)^{\frac{1}{2}}$$

In which, assuming

8 $$\sin Z'' = \frac{\cos \tfrac{1}{2} a (\sin B \sin C)^{\frac{1}{2}}}{\cos \tfrac{1}{2} (B \backsim C)}$$

The formula becomes:

9 $$\sin \tfrac{1}{2} A = \cos \tfrac{1}{2} (B \backsim C) \cos Z''$$

4th Transformation. By substituting the functions of the half angles, in a manner the inverse of that used in the preceding transformation, we obtain, from the same formula as the preceding one, the following results in succession:

$$2 \cos^2 \tfrac{1}{2} a \sin B \sin C = 2 \cos^2 \tfrac{1}{2} A - 2 \sin^2 \tfrac{1}{2} (B \backsim C)$$

Whence

$$\cos^2 \tfrac{1}{2} A = \sin^2 \tfrac{1}{2} (B \backsim C) + \cos^2 \tfrac{1}{2} a \sin B \sin C$$

10 $$\cos \tfrac{1}{2} A = \sin \tfrac{1}{2} (B \backsim C) \left(1 + \frac{\sin B \sin C \cos^2 \tfrac{1}{2} a}{\sin^2 \tfrac{1}{2} (B \backsim C)}\right)^{\frac{1}{2}}$$

And assuming

11 $$\tan Z''' = \frac{\cos \tfrac{1}{2} a (\sin B \sin C)^{\frac{1}{2}}}{\sin \tfrac{1}{2} (B \backsim C)}$$

The formula becomes:

12 $$\cos \tfrac{1}{2} A = \frac{\sin \tfrac{1}{2} (B \backsim C)}{\cos Z'''}$$

5th Transformation. The four preceding transformations, also evidently give two for the tangents; viz.

By dividing formula 3, by formula 6, we obtain:

$$\frac{\sin \frac{1}{2} A}{\cos \frac{1}{2} A} = \tan \frac{1}{2} A = \frac{\cos \frac{1}{2}(B+C)}{\sin \frac{1}{2}(B+C) \cos Z \cos Z'} = \frac{\cot \frac{1}{2}(B+C)}{\cos Z \cos Z'} \qquad 13$$

for which Z, and Z', are determined by No. 2 and 8.

6th Transformation. By division of the formula No. 9, by the formula No. 12, is obtained:

$$\frac{\sin \frac{1}{2} A}{\cos \frac{1}{2} A} = \tan \frac{1}{2} A = \frac{\cos \frac{1}{2}(B \backsim C)}{\sin \frac{1}{2}(B \backsim C)} \cos Z'' \cos Z'''$$

$$= \cot \frac{1}{2}(B \backsim C) \cos Z'' \cos Z''' \qquad 14$$

where Z'', and Z''', are determined by No. 8 and 11, above.

7th Transformation. Taking the original formula, as in problem 4, we obtain, by transformations analogous to those for No. 16 and 18, in the 5th problem, the following results in succession, for two analogous transformations; viz:

$$\cos A = \cos a \sin B \sin C - \cos B \cos C$$

By a substitution, according to B, No. 8, using for sine B, its equal, tangent B cosine B:

$$\cos A = \cos a \tan B \cos B \sin C - \cos B \cos C$$

And assuming

$$\cot y = \cos a \tan B \qquad 15$$

$$\cos A = \cos B (\cot y \sin C - \cos C)$$

$$= \frac{\cos B}{\sin y} (\cos y \sin C - \cos C \sin y)$$

$$\cos A = \frac{\cos B}{\sin y} \sin (C - y) \qquad 16$$

8th Transformation. This is identical with the foregoing, and is thus obtained. If, instead of making use of cotangent y, we assume:

17 $$\tan y' = \cos a \tan B$$

we obtain:

$$\cos A = \cos B\,(\tan y' \sin C - \cos C)$$

$$= \frac{\cos B}{\cos y'}\,(\sin y' \sin C - \cos C \cos y')$$

13 $$\cos A = \frac{-\cos B \cos (y' + C)}{\cos y'}$$

It is necessary to pay strict attention to the proper algebraic signs of $(y' + C)$ that determine the affection of A, which is obtuse when $(y' + C) \angle 90^\circ$, and acute when $(y' + C)$ ⦣ 90°; and also to the algebraic sign of the functions of y' itself, as determined by a, and B.

§ 92. *Problem 7.* Given, two sides and the included angle, to find the remaining angles.

a, b, and C, being given, to find A, and B.

By d, No. 4, we have:

$$\cot \left(\frac{C_{,} \backsim C_{,,}}{2}\right) = \frac{\cot \frac{1}{2} (B + A) \cot \frac{1}{2} C}{\tan \frac{1}{2} (A \backsim B)}$$

By d, No. 5, we have:

$$\cot \left(\frac{C_{,} \backsim C''}{2}\right) = \frac{\sin (a + b) \tan \frac{1}{2} C}{\sin (a \backsim b)}$$

Therefore, by equality:

$$\frac{\cot \frac{1}{2} (A + B) \cot \frac{1}{2} C}{\tan \frac{1}{2} (A \backsim B)} = \frac{\sin (a + b) \tan \frac{1}{2} C}{\sin (a \backsim b)}$$

Which gives the two following equations:

$$\tan \tfrac{1}{2}(A \backsim B) = \frac{\sin (a \backsim b) \cot^2 \frac{1}{2} C}{\sin (a + b) \tan \frac{1}{2}(A + B)}$$

$$\tan \tfrac{1}{2}(A + B) = \frac{\sin (a \backsim b) \cot^2 \frac{1}{2} C}{\sin (a + b) \tan \frac{1}{2}(A \backsim B)}$$

Each of these formulæ still contains the unknown quantity of the other; by comparing each of them with No. 1 of the same series, we obtain:

$$\tan \tfrac{1}{2}(B \backsim A) =$$

$$\frac{\tan \frac{1}{2}(A + B) \tan \frac{1}{2}(b \backsim a)}{\tan \frac{1}{2}(a + b)} = \frac{\sin (a \backsim b) \cot^2 \frac{1}{2} C}{\sin (a + b) \tan \frac{1}{2}(A + B)}$$

$$\tan \tfrac{1}{2}(A + B) =$$

$$\frac{\tan \frac{1}{2}(A \backsim B) \tan \frac{1}{2}(a + b)}{\tan \frac{1}{2}(a \backsim b)} = \frac{\sin (a \backsim b) \cot^2 \frac{1}{2} C}{\sin (a + b) \tan \frac{1}{2}(A \backsim B)}$$

Which furnish two new equations, each of which contains no more than one of the two unknown quantities that are sought, and we can easily deduce from them:

$$\tan^2 \tfrac{1}{2}(A + B) = \frac{\cot^2 \frac{1}{2} C \sin (a \backsim b) \tan \frac{1}{2}(a + b)}{\sin (a + b) \tan \frac{1}{2}(a \backsim b)}$$

$$\tan^2 \tfrac{1}{2}(A \backsim B) = \frac{\cot^2 \frac{1}{2} C \sin (a \backsim b) \tan \frac{1}{2}(a \backsim b)}{\sin (a + b) \tan \frac{1}{2}(a + b)}$$

By a substitution for sin $(a \underset{\backsim}{+} b)$, analogous to Q, No, 1, expressing the tan $= \dfrac{\sin}{\cos}$, and the resulting compensations and extraction of the roots, the final formulæ result:

n

$$\tan \tfrac{1}{2}(A + B) = \cot \tfrac{1}{2} C \frac{\cos \frac{1}{2}(a \backsim b)}{\cos \frac{1}{2}(a + b)} \qquad 1$$

S

2 $$\tan \tfrac{1}{2}(A \backsim B) = \cot \tfrac{1}{2} C \, \frac{\sin \frac{1}{2}(a \backsim b)}{\sin \frac{1}{2}(a + b)}$$

These formulæ, which are easy to calculate, and advantageous, enable us to dispense with the research of others, that might be easily constructed, in which no more than a single angle is determined. They give, of course, directly:

$$A = \tfrac{1}{2}(A + B) + \tfrac{1}{2}(A \backsim B)$$

and $$B = \tfrac{1}{2}(A + B) - \tfrac{1}{2}(A \backsim B)$$

§ 93. *Problem* 8. Given, two angles and the contained side, to find the two remaining sides.

A, *B*, and *c*, being given, to find *a*, and *b*.

The consideration of the formulæ of the same series whence the preceding have been derived, shows that formulæ similar to them may be obtained for this case. But in order to shorten the operation, we shall here proceed by means of the supplementary triangle; writing each of the angles and sides in expressions taken from the supplementary triangle; distinguishing them by accentuation until reduced. When reduced by this method, the formulæ n, No. 1 and 2, become:

$$\tan \frac{180^\circ - a' + 180^\circ - b'}{2}$$
$$= \cot (90^\circ - \tfrac{1}{2} c') \, \frac{\cos \frac{1}{2}\left((180^\circ - A') \backsim (180^\circ - B')\right)}{\cos \frac{1}{2}\left((180^\circ - A') + (180^\circ - B')\right)}$$

$$\tan \tfrac{1}{2}\left((180^\circ - a') \backsim (180^\circ - b')\right)$$
$$= \cot (90^\circ - \tfrac{1}{2} c') \, \frac{\sin \frac{1}{2}\left((180^\circ - A') \backsim (180^\circ - B')\right)}{\sin \frac{1}{2}\left((180^\circ - A') + (180^\circ - B')\right)}$$

Making the compensations which evidently result throughout, considering that $\cos(180^\circ - x) = -\cos x$; that $\cot(90^\circ - \frac{1}{2} c') = \tan \frac{1}{2} c'$, and that $\tan(180^\circ - x) = -\tan x$, and ultimately omitting the accentuation, we have as a final result:

o

$$\tan \tfrac{1}{2}(a+b) = \tan \tfrac{1}{2} c \, \frac{\cos \tfrac{1}{2}(A \backsim B)}{\cos \tfrac{1}{2}(A+B)} \qquad 1$$

$$\tan \tfrac{1}{2}(a \backsim b) = \tan \tfrac{1}{2} c \, \frac{\sin \tfrac{1}{2}(A \backsim B)}{\sin \tfrac{1}{2}(A+B)} \qquad 2$$

The two sides here, are evidently found as the two angles were in the former instance.

$$a = \tfrac{1}{2}(a+b) + \tfrac{1}{2}(a \backsim b)\,; \qquad b = \tfrac{1}{2}(a+b) - \tfrac{1}{2}(a \backsim b)$$

If two sides with the two angles opposite to them are given, the formulæ n, and o, determine with equal ease the third angle and the third side.

§ 94. *Problem* 9. Given, two sides and an angle opposite to one of them, to find the third side.

b, a, and A, being given, to find c.

By f, No. 3, we have:

$$\cos a = \cos c \cos b + \sin b \sin c \cos A$$

This formula has already been transformed, in problem 5, formulæ l, No. 15 to 18, into the two following.

1*st Transformation.* Making

p

$$\tan y = \tan b \cos A \qquad 1$$

we there obtained:

$$\cos a = \frac{\cos b}{\cos y} \cos (c \backsim y)$$

Whence, dividing by $\dfrac{\cos b}{\cos y}$, we obtain for the present problem:

$$\cos (c \backsim y) = \frac{\cos a \cos y}{\cos b} \qquad 2$$

2d Transformation. Making

3 $$\cot y' = \cos A \tan b$$

we there obtained:

$$\cos a = \frac{\cos b}{\sin y'} \sin (c + y')$$

Whence we deduce, as before:

4 $$\sin (c + y') = \frac{\cos a \sin y'}{\cos b}$$

These two formulæ give, each, one of the two possible angles. Only one, however, need be calculated, because this double result is also obtained by taking both the sum, and the difference between the two angles, c, and $c \pm y$; with this understanding, therefore, the two formulæ are identical.

3d Transformation. By a direct analytical treatment of the formula f, No. 3, we may obtain a mutation giving the side c, in a formula analogous to F, No. 1, or 2; that is to say, in two parts; the sum or difference of which will be the side c, sought; this process leads through a quadratic equation, which may be avoided by proceeding as has been done in Plane Trigonometry, *problem* 5.

Making, therefore, according to *figure* 15, *or* 16:

5 $$c = BD \pm AD = x \pm y$$

We have by series b, No. 3:

6 $$\tan AD = \tan b \cos A = \tan x$$

7 and $$\tan BD = \tan a \cos B = \tan y$$

The angle B, is not given directly; but is determinable from the data of the problem; we have by them:

8 $$\sin B = \frac{\sin A \sin b}{\sin a}$$

This angle, *B*, therefore, forms an auxiliary arc in the determination of tangent *y*, by which means both parts of *c* are, therefore, determined by their tangents.

4*th Transformation.* If we express, by means of the value of the tangents obtained in the preceding, the values of the cosines of *x* and *y*, according to the formula D, No. 9, we obtain the following simple expressions, which are extremely easy to calculate. We have, in that case:

$$\cos x = \frac{1}{(1 + \tan^2 x)^{\frac{1}{2}}}$$

and in this, by 6:

$$\tan x = \frac{\sin b \cos A}{\cos b}$$

Thence:

$$\cos x = \frac{1}{\left(1 + \dfrac{\sin^2 b \cos A}{\cos^2 b}\right)^{\frac{1}{2}}}$$

$$= \frac{\cos b}{(\cos^2 b + \sin^2 b \cos^2 A)^{\frac{1}{2}}}$$

$$= \frac{\cos b}{(1 - \sin^2 b \sin^2 A)^{\frac{1}{2}}} \qquad 9$$

And by changing the expression for the value of tangent *y*, in like manner, we obtain: by first substituting in 7, the value of $\cos B = (1 - \sin^2 B)^{\frac{1}{2}}$

$$\tan y = \frac{\sin a}{\cos a} \left(1 - \frac{\sin^2 A \sin^2 b}{\sin^2 a}\right)^{\frac{1}{2}}$$

Substituting this value in the same formula, D, No. 9, we obtain:

$$\cos y = \frac{1}{(1 + \tan^2 y)^{\frac{1}{2}}}$$

$$= \frac{1}{\left(1 + \frac{\sin^2 a}{\cos^2 a}\left(1 - \frac{\sin^2 A \sin^2 b}{\sin^2 a}\right)\right)^{\frac{1}{2}}}$$

$$= \frac{1}{\left(1 + \frac{1}{\cos^2 a}(\sin^2 a - \sin^2 A \sin^2 b)\right)^{\frac{1}{2}}}$$

$$= \frac{\cos a}{(\cos^2 a + \sin^2 a - \sin^2 A \sin^2 b)^{\frac{1}{2}}}$$

10 $$\cos y = \frac{\cos a}{(1 - \sin^2 A \sin^2 b)^{\frac{1}{2}}}$$

These two formulæ have the same denominator; they furnish, consequently, the same auxiliary arc for both; making, therefore:

11 $$\sin Z = \sin A \sin b$$

we finally obtain, for the two formulæ 9 and 10, the definitive values:

12 $$\cos x = \frac{\cos b}{\cos Z}$$

and

13 $$\cos y = \frac{\cos a}{\cos Z}$$

the calculation of which is evidently of the greatest simplicity.

By using D, No. 3, instead of No. 9, somewhat similar expressions are obtained for the sines; but, as they are not so simple as either of the preceding ones, they are not here deduced.

§ 95. *Problem* 10. Given, two angles and a side opposite to one of them; to find the third angle.

Given, A, B, a; to find C.

The general formula for this case is again:

$$\cos C = \sin B \sin A \cos c - \cos B \cos A$$

1st *Transformation.* It is evident that, making use of the *7th transformation of problem* 6, we obtain by simple division of m, No. 16:

Assuming q

$$\cot y = \cos a \tan B$$ 1

$$\sin (C \backsim y) = \frac{\cos A \sin y}{\cos B}$$ 2

2d *Transformation.* By the same process, we obtain from m, No. 18, upon the supposition that:

$$\tan y' = \cos a \tan B$$ 3

$$\cos (C + y') = - \frac{\cos A \cos y'}{\cos B}$$ 4

These two formulæ are evidently under the same predicament as the two corresponding ones in the preceding problem.

3d *Transformation.* To this, what has been said in the 3d *transformation of problem* 9, again applies exactly; for we have here again, by *figure* 15 *and* 16:

$$C = BCD \pm ACD = x \pm y$$ 5

By series b, No. 5, we obtain for this case:

$$\cot x = \cos a \tan B$$ 6

$$\cot y = \cos b \tan A$$ 7

The unknown side b, employed here, is determined according to *problem* 1, as an auxiliary arc for No. 7, thus:

$$\sin b = \frac{\sin a \sin B}{\sin A}$$ 8

4th Transformation. The results of the preceding formula being used, in the same manner as in the 4th transformation of the preceding problem, but applied to D, No. 2, will give for this problem the following transformation, exactly analogous in point of form. We have there:

$$\sin x = \frac{1}{(1 + \cot^2 x)^{\frac{1}{2}}}$$

From No. 6, above:

$$\cot x = \cos a \frac{\sin B}{\cos B}$$

Whence

$$\sin x = \frac{1}{\left(1 + \frac{\cos^2 a \sin^2 B}{\cos^2 B}\right)^{\frac{1}{2}}}$$

$$= \frac{\cos B}{(\cos^2 B + \sin^2 B \cos^2 a)^{\frac{1}{2}}}$$

$$= \frac{\cos B}{(\cos^2 B + \sin^2 B - \sin^2 B \sin^2 a)^{\frac{1}{2}}}$$

9 $$\sin x = \frac{\cos B}{(1 - \sin^2 B \sin^2 a)^{\frac{1}{2}}}$$

And in like manner, after having inserted the auxiliary angle No. 8, we have:

$$\cot y = \frac{\sin A}{\cos A}\left(1 - \frac{\sin^2 a \sin^2 B}{\sin^2 A}\right)^{\frac{1}{2}}$$

$$= \frac{(\sin^2 A - \sin^2 a \sin^2 B)^{\frac{1}{2}}}{\cos A}$$

Whence is obtained, by analogy to D, No. 2:

$$\sin y = \frac{1}{\left(1 + \frac{\sin^2 A - \sin^2 a \sin^2 B}{\cos^2 A}\right)^{\frac{1}{2}}}$$

$$= \frac{\cos A}{(\cos^2 A + \sin^2 A - \sin^2 a \sin^2 B)^{\frac{1}{2}}}$$

$$\sin y = \frac{\cos A}{(1 - \sin^2 a \sin^2 B)^{\frac{1}{2}}} \qquad 10$$

Here, again, the denominators are equal; and the auxiliary arc

$$\sin Z = \sin a \sin B \qquad 11$$

Which gives the final formulæ:

$$\sin x = \frac{\cos B}{\cos Z} \qquad 12$$

and

$$\sin y = \frac{\cos A}{\cos Z} \qquad 13$$

§ 96. *Problem* 11. Given, two angles, and a side opposite to one of them; to find the side contained between these angles.

Given, *B*, *a*, *A*; to find *c*.

The solution of this case depends upon the original formula, f, No. 2.

$$\sin c \cot a + \cos c \cos B = \sin B \cot A$$

1*st Transformation.* By making r

$$\tan x = \cos B \tan a \qquad 1$$

which gives:

$$\cot a = \frac{\cos B \cos x}{\sin x}$$

And substituting it in the equation, it becomes:

$$\frac{\sin c \cos B \cos x}{\sin x} + \cos c \cos B = \sin B \cot A$$

Reducing to a common denominator, and making $\frac{\cos B}{\sin x}$ a common factor:

$$\frac{\cos B}{\sin x}(\sin c \cos x + \cos c \sin x) = \sin B \cot A$$

$$\frac{\cos B \sin (c + x)}{\sin x} = \sin B \cot A$$

And finally:

2 $\sin (c + x) = \tan B \cot A \sin x$

2d Transformation. This is obtained in a manner similar to the foregoing, by making:

3 $\cot x' = \cos B \tan a$

Which gives:

$$\cot a = \frac{\cos B \sin x'}{\cos x'}$$

Which, being substituted in the formula, gives:

$$\frac{\sin c \cos B \sin x'}{\cos x'} + \cos c \cos B = \sin B \cot A$$

Treating this as in the previous case, we obtain successively:

$$\frac{\cos B}{\cos x'}(\sin c \sin x' + \cos c \cos x') = \sin B \cot A$$

$$\frac{\cos B \cos (c \backsim x')}{\cos x'} = \sin B \cot A$$

4 $\cos (c \backsim x') = \tan B \cot A \cos x'$

that are two formulæ, in the same predicament as q, No. 2 and 4.

3d *Transformation.* This is made in a manner analogous to that in *problem* 9, and equally avoids the quadratic equation, by which it could be obtained from f, No. 3, or e, No. 3; taking, according to *figure* 15 *and* 16, the segments of the side:

$$c = BD \pm AD = x \pm y \qquad 5$$

And taking, in b, No. 4, the value of these parts, we obtain:

$$\tan x = \tan a \cos B \qquad 6$$

$$\tan y = \tan b \cos A \qquad 7$$

The side *b*, which is here supposed to be given, and must therefore be determined as an auxiliary angle from the data of the problem, is:

$$\sin b = \frac{\sin B \sin a}{\sin A} \qquad 8$$

By which, again, all parts are solved, and the result calculable by logarithms.

4*th Transformation.* From this last formula can be deduced another, in the same way as in the two preceding problems, for the same two sections *x*, and *y*, of the side *c*; thus:

Expressing the cosine by D, No. 9, we have:

$$\cos x = \frac{1}{(1 + \tan^2 x)^{\frac{1}{2}}}$$

Which gives here, by substituting the value of tangent *x*, that we have just obtained:

$$\cos x = \frac{1}{(1 + \tan^2 a \cos^2 B)^{\frac{1}{2}}}$$

$$= \frac{\cos a}{(\cos^2 a + \cos^2 B \sin^2 a)^{\frac{1}{2}}}$$

9 $$\cos x = \frac{\cos a}{(1 - \sin^2 a \sin^2 B)^{\frac{1}{2}}}$$

And taking D, No. 3, to express sine y in terms of tangent y, we have:

$$\sin y = \frac{\tan y}{(1 + \tan^2 y)^{\frac{1}{2}}}$$

And expressing tangent b by the auxiliary arc, we have:

$$\tan b = \frac{\sin a \sin B}{\sin A \left(1 - \dfrac{\sin^2 a \sin^2 B}{\sin^2 A}\right)^{\frac{1}{2}}}$$

$$= \frac{\sin a \sin B}{(\sin^2 A - \sin^2 a \sin^2 B)^{\frac{1}{2}}}$$

Whence

$$\tan y = \frac{\cos A \sin a \sin B}{(\sin^2 A - \sin^2 a \sin^2 B)^{\frac{1}{2}}}$$

and

$$\sin y = \frac{\dfrac{\cos A \sin a \sin B}{(\sin^2 A - \sin^2 a \sin^2 B)^{\frac{1}{2}}}}{\left(1 + \dfrac{\sin^2 a \sin^2 B \cos^2 A}{\sin^2 A - \sin^2 a \sin^2 B}\right)^{\frac{1}{2}}}$$

Bringing the denominator to a common denominator, and compensating, in numerator and denominator:

$$\sin y = \frac{\cos A \sin a \sin B}{(\sin^2 A - \sin^2 a \sin^2 B + \sin^2 a \sin^2 B \cos^2 A)^{\frac{1}{2}}}$$

$$= \frac{\cos A \sin a \sin B}{\left(\sin^2 A - \sin^2 a \sin^2 B\,(1 - \cos^2 A)\right)^{\frac{1}{2}}}$$

$$\sin y = \frac{\cos A \sin a \sin B}{\sin A\,(1 - \sin^2 a \sin^2 B)^{\frac{1}{2}}}$$

$$= \frac{\cot A \sin a \sin B}{(1 - \sin^2 a \sin^2 B)^{\frac{1}{2}}} \qquad 10$$

Here we have again, for the determination of x and y, the same denominator, and therefore the same auxiliary arc. Therefore, making

$$\sin Z = \sin a \sin B \qquad 11$$

we obtain finally the two following formulæ for calculation:

$$\cos x = \frac{\cos a}{\cos Z} \qquad 12$$

and

$$\sin y = \frac{\cot A \sin a \sin B}{\cos Z} = \cot A \tan Z \qquad 13$$

Of these two expressions for sine y, the first will be found shorter in the actual calculation, because it is easier to write sine a sine B, twice, and use the same auxiliary arc, than to take two different auxiliary angles.

§ 97. *Problem* 12. Given, two sides and an angle opposite to one of them; to find the contained angle.

Given, A, b, a; to find C.

By f, No. 2, we have:

s

$$\sin C \cot A + \cos b \cos C = \sin b \cot a \qquad 1$$

1*st Transformation* is obtained as in the preceding problem, by making

$$\tan x = \cos b \tan A$$

Which gives:

$$\cot A = \frac{\cos b \cos x}{\sin x}$$

The equation becomes, by this substitution:

$$\frac{\cos C \cos b \cos x}{\sin x} + \cos b \cos C = \sin b \cot a$$

Reducing to a common denominator, and making $\frac{\cos b}{\sin x}$ a common factor, gives:

$$\frac{\cos b}{\sin x}(\sin C \cos x + \cos C \sin x) = \sin b \cot a$$

or

$$\frac{\cos b \sin (C + x)}{\sin x} = \sin b \cot a$$

Whence, finally:

2 $\sin (C + x) = \cot a \tan b \sin x$

2d *Transformation.* A formula corresponding to the preceding is obtained, by making

3 $\cot x' = \cos b \tan A$

And following the same process in the reduction of this as in the preceding, the final formula will become:

4 $\cos (C \backsim x') = \tan b \cot a \cos x'$

These two formulæ are again to be considered and treated as the two, q, No. 2 and 4.

3d *Transformation.* Here, as in problem 10, expressing the two segments of the angle C, we obtain by means of b, No. 5, for each of them, simple formulæ for logarithmic calculation; thus:

5 $C = BCD \pm ACD = x \pm y$

We have by b, No. 5:

6 $\cot x = \cos a \tan B$

and

7 $\cot y = \cos b \tan A$

Where B is to be determined from the data of the problem, and used as an auxiliary angle; thus:

$$\sin B = \frac{\sin b \sin A}{\sin a} \qquad 8$$

4th Transformation. We may here again apply with advantage, the transformations given in formulæ 6, and 7, by the aid of D, No. 14 and 21.

We have from 8, the value:

$$\tan B = \frac{\sin b \sin A}{\sin a \left(1 - \frac{\sin^2 b \sin^2 A}{\sin^2 a}\right)^{\frac{1}{2}}}$$

$$= \frac{\sin b \sin A}{(\sin^2 a - \sin^2 b \sin^2 A)^{\frac{1}{2}}}.$$

Whence

$$\cot x = \cos a \tan B = \frac{\cos a \sin b \sin A}{(\sin^2 a - \sin^2 b \sin^2 A)^{\frac{1}{2}}}$$

By D, No. 14, we have:

$$\cos x = \frac{\cot x}{(1 + \cot^2 x)^{\frac{1}{2}}} = \frac{\frac{\cos a \sin b \sin A}{(\sin^2 a - \sin^2 b \sin^2 A)^{\frac{1}{2}}}}{\left(1 + \frac{\sin^2 b \cos^2 a \sin^2 A}{\sin^2 a - \sin^2 b \sin^2 A}\right)^{\frac{1}{2}}}$$

$$= \frac{\cos a \sin b \sin A}{(\sin^2 a - \sin^2 b \sin^2 A + \sin^2 b \cos^2 a \sin^2 A)^{\frac{1}{2}}}$$

$$= \frac{\cos a \sin b \sin A}{\left(\sin^2 a - \sin^2 b \sin^2 A\,(1 - \cos^2 a)\right)^{\frac{1}{2}}}$$

$$= \frac{\cos a \sin b \sin A}{\sin a\,(1 - \sin^2 b \sin^2 A)^{\frac{1}{2}}}$$

9 $$\cos x = \frac{\cot a \sin b \sin A}{(1 - \sin^2 b \sin^2 A)^{\frac{1}{2}}}$$

From D, No. 2, we have for

$$\sin y = \frac{1}{(1 + \cot^2 y)^{\frac{1}{2}}} = \frac{1}{\left(1 + \frac{\cos^2 b \sin^2 A}{\cos^2 A}\right)^{\frac{1}{2}}}$$

$$= \frac{\cos A}{(\cos^2 A + \cos^2 b \sin^2 A)^{\frac{1}{2}}}$$

$$= \frac{\cos A}{(\cos^2 A + \sin^2 A - \sin^2 A \sin^2 b)^{\frac{1}{2}}}$$

10 $$\sin y = \frac{\cos A}{(1 - \sin^2 A \sin^2 b)^{\frac{1}{2}}}$$

The two formulæ 9 and 10, thus obtained, have again th
property of having the same denominator; therefore, makin
use of the same auxiliary angle, namely:

11 $$\sin Z = \sin A \sin b$$

the final formulæ for calculation become:

12 $$\cos x = \frac{\cot a \sin b \sin A}{\cos Z} = \cot a \tan Z$$

and

13 $$\sin y = \frac{\cos A}{\cos Z}$$

What has been said in relation to r, No. 13, also applie
here to No. 12.

§ 98. We have thus obtained for each of the cases of ob
lique angled spherical trigonometry, a variety of formulæ, c
easy calculation by logarithms; as it may be useful in prac

tice to have complete formulæ for every case, we have considered it proper to enter into these details in this part of the treatise, in order that a choice may be made among the formulæ of such as may best suit in any individual case, and afford the greatest accuracy. A skilful calculator will also find in them a check upon his own numeric operations; for he may at the same time calculate by means of two different formulæ; for which purpose he will choose such as are most easily used simultaneously, in consequence of their only differing from each other in the employment of different trigonometric functions of the same elements.

These formulæ all concur in showing: that, in Spherical Trigonometry, under equal circumstances, the different parts equally depend upon their data for their form of combination; the part sought may be either a side or an angle; so that there are in truth only six forms of this mutual dependence of the parts, which differ only in the details of signs, and occasional changes between sines and cosines, or tangents and cotangents.

It may be easily conceived, when we consider the multitude of analytic formulæ that may be deduced from the nature of the trigonometric functions: that other formulæ and transformations, besides those here presented, are possible, as well as other methods; but those here given are, in general, the most direct, and most accurate, and are consequently of most frequent use.

In all the above transformations, the formulæ from which they originate, or any particular operation performed, which may not be immediately evident, has been quoted and referred to, and, in addition, the aim of any operation, and the intended mode, has generally been quoted before the operation; but it has been uniformly supposed, as stated in the beginning, that series B and C were known, as it is supposed that any arithmetician knows his multiplication table; though it is not necessary to learn them by heart, for the proper study of the first elements, and practice, will very soon make them as familiar as the multiplication table is to calculators.

§ 99. In order to decide the doubtful cases, as indicated by the formulæ, or the nature of the cases, we may observe a few general rules.

1. That in the formulæ 1, 2, 3, and 4, of the problems 9, 10, 11, and 12, the tangent or cotangent of the auxiliary arc, and the cosines of the other parts, may change sign; which, therefore, must be attended to.

2. That by never employing a triangle with a side or angle exceeding 80°, the results that would lead to such a side or angle are of course excluded.

3. That in every triangle, the greatest sides and greatest angles are opposite, the least side to the least angle, and the mean to the mean.

4. The principle, that the sum of the three sides of the triangle is always less, and the sum of the three angles, always more than four right angles, sometimes gives another criterion to judge in the case; as well as: that the sum of the angles shall not exceed six right angles.

5. The circumstances of a given case rarely leave room for doubt in the decision. It has already been observed: that wherever the case is not doubtful by nature, the formulæ giving half angles are the most advantageous, in this, as well as in other respects.

FORMULÆ for the Surface of Triangles.

Series.	*No.*	*Data.*	*Sought.*	*Analytical Formulæ for Calculation.*	AUXILIARY ARCS OR ELEMENTS.	
					Auxiliary.	*Analytic Formulæ.*
Z	1 and 2	B, a, C	S	$S = \frac{a^2 \sin b \sin C}{2 \sin (B + C)} = \frac{a^2}{2 (\cot B + \cot C)}$		
	3	a, C, b	S	$S = \frac{a b \sin C}{2}$		
	5	a, b, c	S	$S = (p (p - a) (p - b) (p - c))^{\frac{1}{2}}$	p	$= \frac{a + b + c}{2}$
	6	b, C, c	S	$S = \frac{b^2 \cos C \sin C}{2} \pm \frac{c . b \sin C}{2} \left(1 - \frac{b^2 \sin^2 C}{c^2}\right)^{\frac{1}{2}}$		

TABLE of the FORMULÆ of Oblique Angled Spherical Trigonometry.

Data.	Sought.	Series.	No.	Function determined.	Analytical Formulæ for the part sought.	AUXILIARY ARCS OR ELEMENTS.		
						No.	Auxiliary.	Formulæ for the Auxiliary.
c, B, b	C	g		$\sin C =$	$\dfrac{\sin B \sin c}{\sin b}$			
B, c, C	b	h		$\sin b =$	$\dfrac{\sin B \sin c}{\sin C}$			
a, b, c	A	i	1	$\sin \frac{1}{2} A =$	$\left(\dfrac{\sin (p - c) \sin (p - b)}{\sin c \sin b}\right)^{\frac{1}{2}}$		$p =$	$\dfrac{a + b + c}{2}$
			2	$\cos \frac{1}{2} A =$	$\left(\dfrac{\sin p \sin (p - a)}{\sin b \sin c}\right)^{\frac{1}{2}}$			
			3	$\tan \frac{1}{2} A =$	$\left(\dfrac{\sin (p - c) \sin (p - b)}{\sin p \sin (p - a)}\right)^{\frac{1}{2}}$			
A, B, C	a	k	1	$\sin \frac{1}{2} a =$	$\left(\dfrac{\cos P \cos (P - A)}{\sin B \sin C}\right)^{\frac{1}{2}}$		$P =$	$\dfrac{A + B + C}{2}$
			2	$\cos \frac{1}{2} a =$	$\left(\dfrac{\cos (P - C) \cos (P - B)}{\sin B \sin C}\right)^{\frac{1}{2}}$			

TABLE of the FORMULÆ of Oblique Angled Spherical Trigonometry, Continued.

Data.	Sought.	Series.	No.	Function determined.	Anlyatical Formulæ for the part sought.	AUXILIARY ARCS OR ELEMENTS.		
						No.	Auxiliary.	Formulæ for the Auxiliary.
A, B, C	a	k	3	$\tan \frac{1}{2} a$	$= \left(\frac{\cos P \cos (P - A)}{\cos (P - C) \cos (P - B)} \right)^{\frac{1}{2}}$			
A, c, b	a	1	3	$\sin \frac{1}{2} a$	$= \frac{\sin \frac{1}{2} (c \backsim b)}{\cos Z}$	2	$\tan Z$	$= \frac{\sin \frac{1}{2} A}{\sin \frac{1}{2} (c \backsim b)} (\sin b \sin c)^{\frac{1}{2}}$
			6	$\cos \frac{1}{2} a$	$= \cos \frac{1}{2} (c \backsim b) \cos Z'$	5	$\sin Z'$	$= \frac{\sin \frac{1}{2} A}{\cos \frac{1}{2} (c \backsim b)} (\sin b \sin c)^{\frac{1}{2}}$
			9	$\sin \frac{1}{2} a$	$= \sin \frac{1}{2} (c + b) \cos Z''$	8	$\sin Z''$	$= \frac{\cos \frac{1}{2} A}{\sin \frac{1}{2} (c + b)} (\sin b \sin c)^{\frac{1}{2}}$
			12	$\cos \frac{1}{2} a$	$= \frac{\cos \frac{1}{2} (c + b)}{\cos Z'''}$	11	$\tan Z'''$	$= \frac{\cos \frac{1}{2} A}{\cos \frac{1}{2} (c + b)} (\sin b \sin c)^{\frac{1}{2}}$
			13	$\tan \frac{1}{2} a$	$= \frac{\tan \frac{1}{2} (c \backsim b)}{\cos Z \cos Z'}$			
			14	$\tan \frac{1}{2} a$	$= \tan \frac{1}{2} (c + b) \cos Z'' \cos Z'''$			
			16	$\cos a$	$= \frac{\cos b \cos (c \backsim y)}{\cos y}$	15	$\tan y$	$= \cos A \tan b$

Data.	*Sought.*	*Series.*	*No.*	*Function determined.*	*Analytical Formulæ for the part sought.*	AUXILIARY ARCS OR ELEMENTS. *No.*	*Auxiliary.*	*Formulæ for the Auxiliary.*
A, c, b	a	l	18	$\cos a$	$= \dfrac{\cos b \sin (c + y')}{\sin y'}$	17	$\cot y'$	$= \cos A \tan b$
B, a, C	A	m	3	$\sin \frac{1}{2} A$	$= \dfrac{\cos \frac{1}{2} (B + C)}{\cos Z}$	2	$\tan Z$	$= \dfrac{\sin \frac{1}{2} a}{\cos \frac{1}{2} (B + C)} (\sin B \sin C)^{\frac{1}{2}}$
			6	$\cos \frac{1}{2} A$	$= \sin \frac{1}{2} (B + C) \cos Z'$	5	$\sin Z'$	$= \dfrac{\sin \frac{1}{2} a}{\sin \frac{1}{2} (B + C)} (\sin B \sin C)^{\frac{1}{2}}$
			9	$\sin \frac{1}{2} A$	$= \cos \frac{1}{2} (B \backsim C) \cos Z''$	8	$\sin Z''$	$= \dfrac{\cos \frac{1}{2} a}{\cos \frac{1}{2} (B \backsim C)} (\sin B \sin C)^{\frac{1}{2}}$
			12	$\cos \frac{1}{2} A$	$= \dfrac{\sin \frac{1}{2} (B \backsim C)}{\cos Z'''}$	11	$\tan Z'''$	$= \dfrac{\cos \frac{1}{2} a}{\sin \frac{1}{2} (B \backsim C)} (\sin B \sin C)^{\frac{1}{2}}$
			13	$\tan \frac{1}{2} A$	$= \dfrac{\cot \frac{1}{2} (B + C)}{\cos Z \cos Z'}$			
			14	$\tan \frac{1}{2} A$	$= \cot \frac{1}{2} (B \backsim C) \cos Z'' \cos Z'''$			
			16	$\cos A$	$= \dfrac{\cos B \sin (C - y)}{\sin y}$	15	$\cot y$	$= \cos a \tan B$

TABLE of the FORMULÆ of Oblique Angled Spherical Trigonometry, Continued.

Data.	Sought.	Series.	No.	Function determined	Analytical Formulæ for the part sought.	AUXILIARY ARCS OR ELEMENTS.		
						No.	Auxiliary.	Formulæ for the Auxiliary.
B, a, C	A	m	18	$\cos A$	$= \dfrac{-\sin B \cos (C + y')}{\cos y'}$	17	$\tan y'$	$= \cos a \tan B$
C, a, b	A, & B	n	1	$\tan \frac{1}{2} (A + B)$	$= \cot \frac{1}{2} C \dfrac{\cos \frac{1}{2} (a \backsim b)}{\cos \frac{1}{2} (a + b)}$			
			2	$\tan \frac{1}{2} (A \backsim B)$	$= \cot \frac{1}{2} C \dfrac{\sin \frac{1}{2} (a \backsim b)}{\sin \frac{1}{2} (a + b)}$			
A, c, B	a, & b	o	1	$\tan \frac{1}{2} (a + b)$	$= \tan \frac{1}{2} c \dfrac{\cos \frac{1}{2} (B \backsim A)}{\cos \frac{1}{2} (A + B)}$			
			2	$\tan \frac{1}{2} (a \backsim b)$	$= \tan \frac{1}{2} c \dfrac{\sin \frac{1}{2} (A \backsim B)}{\sin \frac{1}{2} (A + B)}$			
a, B, b	c	p	2	$\cos (c \backsim y)$	$= \dfrac{\cos a \cos y}{\cos b}$	1	$\tan y$	$= \cos A \tan b$

Data.	Sought.	Series.	No.	Function determined.	Analytical Formulæ for the part sought.	AUXILIARY ARCS OR ELEMENTS.		
						No.	Auxiliary.	Formulæ for the Auxiliary.
a, A, b	c	p	4	$\sin(c + y')$	$= \dfrac{\cos a \sin y'}{\cos b}$	3	$\cot y'$	$= \cos A \tan b$
			5	c	$= x \underset{\backsim}{+} y$	8	$\sin B$	$= \dfrac{\sin A \sin b}{\sin a}$
			6	$\tan x$	$= \cos A \tan b$			
			7	$\tan y$	$= \tan a \cos B$			
			12	$\cos x$	$= \dfrac{\cos b}{\cos Z}$	11	$\sin Z$	$= \sin A \sin b$
			13	$\cos y$	$= \dfrac{\cos a}{\cos Z}$			
B, a, A	C	q	2	$\cos(C \backsim y)$	$= \dfrac{\cos A \sin y}{\cos B}$	1	$\cot y$	$= \tan B \cos a$
			4	$\cos(C + y')$	$= \dfrac{\cos A \cos y'}{\cos B}$	3	$\tan y'$	$= \tan B \cos a$

TABLE of the FORMULÆ of Oblique Angled Spherical Trigonometry, Continued.

Data.	Sought.	Series.	No.	Function determined.	Analytical Formulæ for the part sought.	Auxiliary Arcs or Elements. No.	Auxiliary.	Formulæ for the Auxiliary.
B, a, A	C	q	5	C	$= x \overset{+}{\backsim} y$	8	$\sin b$	$= \dfrac{\sin a \sin B}{\sin A}$
			6	$\cot x$	$= \cos A \tan B$			
			7	$\cot y$	$= \tan A \cos b$			
			12	$\sin x$	$= \dfrac{\cos B}{\cos Z}$	11	$\sin Z$	$= \sin a \sin B$
			13	$\sin y$	$= \dfrac{\cos A}{\cos Z}$			
B, a, C	c	r	2	$\sin (c + x)$	$= \tan B \cot A \sin x$	1	$\tan x$	$= \cos B \tan a$
			4	$\cos (c \backsim x)$	$= \tan B \cot A \cos x'$	3	$\cot x'$	$= \cos B \tan a$
			5	c	$= x \overset{+}{\backsim} y$	8	$\sin b$	$= \dfrac{\sin B \sin a}{\sin A}$
			6	$\tan x$	$= \tan a \cos B$			
			7	$\tan y$	$= \tan b \cos A$			

Data.	Sought.	Series.	No.	Function determined.	Analytical Formulæ for the part sought.	AUXILIARY ARCS OR ELEMENTS.		
						No.	Auxiliary.	Formulæ for the Auxiliary.
B, a, C	c	r	12		$\cos x = \dfrac{\cos a}{\cos Z}$	11	$\sin Z$	$= \sin a \sin B$
			13		$\sin y = \dfrac{\cot A \sin a \sin B}{\cos Z}$			
b, a, A	C	s	2	$\sin(C + x)$	$= \cot a \tan b \sin x$	1	$\tan x$	$= \cos b \tan A$
			4	$\cos(C \backsim x')$	$= \tan b \cot a \cos x'$	3	$\cot x'$	$= \cos b \tan A$
			5		$C = x \underset{\backsim}{+} y$	8	$\sin B$	$= \dfrac{\sin A \sin b}{\sin a}$
			6		$\cot x = \cos a \tan B$			
			7		$\cot y = \cos b \tan A$			
			12		$\cos x = \dfrac{\cot a \sin b \sin A}{\cos Z}$	11	$\sin Z$	$= \sin b \sin A$
			13		$\sin y = \dfrac{\cos A}{\cos Z}$			

PART IV.

PRINCIPLES AND EXAMPLES OF THE PRACTICAL CALCULATIONS OF TRIGONOMETRY.

CHAPTER I.

General Principles of the Calculations.

§ 100. It has already been said, that order, and appropriate arrangement, are qualities indispensable in all calculations; but trigonometric calculations have more especially need of them. There are besides particular methods, that are of special use in such calculations, although they are applicable in a greater or less degree to all.

§ 101. We have seen that the formulæ have been transformed into such as are adapted to the use of logarithms, from the elements to the final result. In order to obtain this object, recourse has frequently been had to what are called auxiliary angles. It must have been observed, that, by means of these, we are enabled to make use of the properties, or rather the different relations of the elementary trigonometric functions, as calculations already made, in which the relative proportion of the variation of these trigonometric functions, is all that remains to be calculated.

§ 102. There is moreover another artifice, that contributes in a high degree to uniformity in the arrangement of the calculation; and it is astonishing, that this has so frequently, and for so long a time, been neglected, although pointed out by Napier, the inventor of logarithms, himself, in his Canon Mirificus. It consists in employing the arithmetical complements of the logarithms; by which the final calculation of a result, depending upon any number of logarithms whatsoever, is reduced to a simple addition.

The general principle of this method may be explained in the following manner.

If from any number whatever, say 783192, we wish to subtract an other, say 639178, it is evident that, if we subtract the latter from the round number of the unit of the next higher denomination of the decimal scale of notation, and add the remainder to the first number, we shall have the same result as if we had made the subtraction; with this difference: that we must reject the unit of the next higher denomination in the decimal scale, that has been thus introduced. In our example we have for the number resulting from the subtraction, which is called the arithmetical complement: 360822
which being added to the first number, or . . . 783192

we have for the sum 144014
after rejecting the unit of the denomination next higher, as above directed; and this is in fact the difference of the two numbers taken as an example.

The use of this method would in ordinary calculations demand a strict attention to the effect of the next higher decimal denomination introduced; as, for instance, if from 379126
we had to subtract 5492, using the complement,
we have to add 4508

which gives in result 373634
where a unit of the fifth denomination, which has been introduced in the complement, is to be rejected, it being the denomination next higher than the highest denomination in the ubstracting number.

§ 103. In logarithmic calculation this rejection becomes merely mechanical; for in them we have always the same number of figures, and the characteristic can never be uncertain to the extent of ten; for this would occasion a difference in the result of ten places of figures in the natural number; a mistake that cannot arise in any given case; and if the result were to be a trigonometric function, it would become an impossible one.

In order then to apply this method in Trigonometry, we always assume a characteristic = 10, from which we deduct the logarithm that is to be subtracted; the complement thus obtained is then added to the logarithm whence the former was to be subtracted. This subtraction from a characteristic = 10 is easily made, as well from right to left, as from left to right, which order may be most convenient in writing; to do this, we suppose the last number to the right to be 10, and all the others 9, and take their complements accordingly; thus each number obtained is the complement to 9 of its corresponding number, except the last on the right, which is the complement to 10; for this, being the first and lowest denomination, borrows from the next higher one a unit, which makes it become = 10, and this same borrowing, extending throughout the series to the characteristic 10, makes all the others, and this characteristic itself, become 9.

To give an example in logarithms, let it be required to
subtract from logarithm 5,3714298
the logarithm 3,2910463

The arithmetical complement of the latter is . 6.7039537
When this is added to the first, the result, after
rejecting 10 from the characteristic, is . . . 2,0803835
which is exactly equal to the difference between the two given logarithms; and having a 10 to reject, and retaining the characteristic 2, gives three significative figures to the whole numbers, the rest being decimals.

Let us take for a second example, one in which the result does not afford a 10 to be rejected, and which is therefore a

proper fraction. Say that from the logarithm 2,7863214
we had to subtract the logarithm 6,2491308

The complement of which would be 3.7508692

Adding this complement to the first log. we obtain 6.5371906
which not furnishing a 10 in the characteristic to be rejected, indicates it to be the logarithm of a decimal fraction; and the characteristic being 6, indicates that the first significative figure of the corresponding decimal fraction is of the fourth place of decimals, or has before it 0,000.

§ 104. This method is besides already introduced in the logarithms of the trigonometric functions; we have there an augmentation of ten units in the characteristic, which corresponds to an assumed radius of 10,000,000,000, instead of unity; which last would make all the trigonometric functions decimals, and their logarithms consequently negative, a result which this system is intended to avoid. This higher characteristic is rejected in the results, as we shall hereafter see, and by that, the method of calculation has only one system.

In order to render the means of ascertaining the number of these supernumerary tens in the characteristic, easy by mere inspection, it is customary to place a simple point (.) after the characteristics that are augmented by 10; and a comma (,) after the characteristic of the logarithms of natural numbers that are not complements. It results from this:—That the number which corresponds to a logarithm whose characteristic is 9, or a less number, with a (.), is a decimal fraction. In order to determine its value, or, which is the same thing, the place of its first significative number, it is to be observed: that the characteristic 10, which corresponds to 0, would give the unit place, and therefore the number which 9 represents would begin with the first decimal place, or tenths; the decimal number whose characteristic is 8, would begin with the second decimal place, or hundredths, and so on, descending in the scale; so that the complement to 10 of the characteristic will indicate the place of decimals held by the first effective figure; the preceding places and the unit place being always filled up with 0; for it is proper to begin every decimal number at the unit place of whole numbers, as well in the case of decimals as in that of whole numbers; for to begin with a (,) or a (.) renders it too easy to mistake this mark as an interpunctuation from the preceding phrase.

There are authors who make use of negative characteristics, which are the complements to 10 of the above arithmetic complements, leaving the logarithms themselves positive; but their use is not only embarrassing, as all additions of positive and negative quantities in the same sum are, but it leads to mistakes in the operation; they are therefore to be rejected.

§ 105. It has been seen, that the formulæ frequently require combinations of the elements by addition and subtraction, in order to obtain numbers or angles, whose logarithms or trigonometric functions (in logarithms) are to be employed in the calculation; a certain order in their arrangement is necessary in order to shorten the calculation itself, and render its verification easy.

In this arrangement all repetition is naturally avoided; if the logarithms serve for several results that are equally the objects of research, they are written in such a way as to be easily added to each of the other logarithms that affect them in the different results; and of the whole is made a single example of calculation, whose parts are added alternately to obtain the respective results.

§ 106. The logarithmic tables that are of most frequent use, have generally seven places of decimals; the degree of exactness obtained by this number of decimals is sufficient for almost every kind of practical calculation. For special purposes there are tables that have ten, and even fifteen, places of decimals; while in cases that require less exactitude, or when the number sought has but few figures, we may be satisfied with using no more than five places of figures. It is to fit them in case of need for this double purpose, that the tables of Callet have a point after the fifth decimal, which saves the attention to, or counting of, the numbers of decimals that are used; but no attention is paid to this point when seven places of decimals are employed. (I may remark here also, that the same tables that give the logarithms of trigonometric functions to every ten seconds, with the differences simply, and are therefore adapted to decimal multiplication, are more convenient, in accurate calculations, where decimals of seconds are used, than the great tables of Taylor giving these logarithms for every second without any differences, which of course must be first obtained, before proportional parts can be taken.)

As for the manner of using logarithmic and trigonometric tables, taking proportional parts, &c. reference must be had to the instructions given upon this subject with every logarithmic table; it would be here a useless repetition, and a description of the several artifices that facilitate their use would be too long if given in detail; attention and reflection in practice will teach them to every able calculator.

§ 107. Let us now proceed to the examples of the calculations themselves. Instead of any explanation that would interrupt the course of calculation, the logarithms will be marked by certain letters, and the results by the algebraic expressions of the operation that these quantities have under-undergone, wherever there is a double operation, otherwise it is supposed, that the sum of the logarithm is taken, as far as not separated by a line. We shall besides point out the data, and place at the top of each calculation the analytical formulæ to be executed, with a reference to the series and number in the body of the work.

CHAPTER II.

Calculations of Plane Trigonometry.

§ 108. **The** following examples may suffice for the calculations of *Right Angled Plane Trigonometry;* as all the other cases give similar processes.

In the right angled plane triangle *ABC*, *figure* 1, given, d, and h, and $A = \llcorner B$; to find sine B.

Formula A, No. 1. $\dfrac{d}{h} = \sin B$

$$\begin{array}{lll} d = 758{,}3 & \log & = 2{,}8798411 \\ h = 1935{,}5 & A:C:\log & = 6.7132068 \\ \hline B = 23°.\ 03'.\ 54'',7 & \log\sin & = 9.5930479 \end{array}$$

Given, h, and B; to find d, and k.

Formula A, No. 1 and 2.

$$d = h \sin B \quad ; \quad k = h \cos B$$

$$\begin{array}{llll} h = 2235{,}0 & \log & = 3{,}3492755 & = x \\ B = 16°.\ 23'.\ 46''. & \log \begin{cases} \sin \\ \cos \end{cases} & \begin{array}{l} = 9.4506865 \\ = 9.9819694 \end{array} & \begin{array}{l} = y \\ = z \end{array} \\ \hline d = 630{,}89 & \log & = 2{,}7999620 & = x + y \\ k = 2144{,}10 & \log & = 3{,}3312449 & = x + z \end{array}$$

Given, d, and k; to find tangent B.

Formula A, No. 3. $\dfrac{d}{k} = \tan B$

$$\begin{array}{lll} d = 31462, & \log & = 4{,}4977863 \\ k = 94723, & A:C:\log & = 5.0235446 \\ \hline B = 18°.\ 22'.\ 25'',6 & & = 9.5213309 \end{array}$$

Given, d, and B; to find h.

Formula A, No. 1.

$$\frac{d}{h} = \sin B\ ; \quad \text{gives } h = \frac{d}{\sin B}$$

$$\begin{array}{lll}
d = 630{,}89 & \log & = 2{,}7999620 \\
B = 16^\circ.\ 23'.\ 46'' & A : C : \log \sin & = \underline{0.5493135} \\
h = 2235{,}0 & \log & = 3{,}3492755
\end{array}$$

§ 109. The calculations of oblique angled plane triangles will follow here in the order of the problems; and are applied to a triangle, *ABC*, *figure* 6, *or* 7, whose sides, *a*, *b*, *c*, are respectively opposite to the angles of the same name.

§ 110. *Problem* 1. Given, B, C, a; to find b, and c.

Formula Y, No. 1.

$$b = \frac{a \sin B}{\sin A}\ ; \quad c = \frac{a \sin C}{\sin A}$$

$$\begin{array}{llll}
a = 3745{,}8 & \log & = 3{,}5735446 & = x \\
A = 61^\circ\ 54'\ 25'' & A : C : \log \sin & = 0.0544409 & = y \\
B = 59.\ 58.\ 40. & \log : \sin & = 9.9374334 & = z \\
C = 58.\ 06.\ 55. & \log : \sin & = 9.9289653 & = w \\
\hline
b = 3676{,}37 & \log & = 3{,}5654189 & = x + y + z \\
c = 3605{,}38 & \log & = 3{,}5569508 & = x + y + w
\end{array}$$

§ 111. *Problem 2.* Given, a, b, C; to find A, and B.

Formula Y, No. 2.

$$\tan\tfrac{1}{2}(A \sim B) = \cot\tfrac{1}{2}C\,\frac{a \sim b}{a+b}$$

$$\begin{aligned}
a &= 4901{,}6 \\
b &= \underline{3620{,}25} \\
a+b &= 8521{,}85 & A:C:\lg &= 6.0694682 \\
a \sim b &= 1281{,}35 & \lg &= 3.1076677 \\
\tfrac{1}{2}C &= 20^\circ\,24'\,10'' & \lg\cot &= \underline{0.4295133} \\
\tfrac{1}{2}(A \sim B) &= 22.\,00.\,39{,}2 & \tan &= 3.6066472 \\
90^\circ - \tfrac{1}{2}C &= \underline{69.\,35.\,50.} \\
A &= 91.\,36.\,29{,}2 \\
B &= 47.\,35.\,10{,}8
\end{aligned}$$

Given, logarithm a, logarithm b, and C; to find A, and B.

Formula Y, No. 3 and 4.

$$\frac{b}{a} = \tan Z \quad ; \quad \tan\tfrac{1}{2}(A \sim B) = \cot\tfrac{1}{2}C\tan(45^\circ - Z)$$

$$\begin{aligned}
&& \lg b &= 3{,}4720537 \\
&& A:C:\lg a &= \underline{6.4833417} \\
z &= 42^\circ\,03'\,46''{,}2 & \lg\tan z &= 9.9553954 \\
& \underline{45.} \\
(45^\circ - z) &= 2.\,56.\,13{,}8 & \lg\tan &= 8.7101908 \\
\tfrac{1}{2}C &= 35.\,17.\,00 & \lg\tan &= \underline{0.1502104} \\
\tfrac{1}{2}(A \sim B) &= 4.\,08.\,50{,}2 & \tan &= 8.8604012 \\
90^\circ - \tfrac{1}{2}C &= \underline{54.\,43.\,00} \\
A &= 58.\,51.\,50{,}2 \\
B &= 50.\,34.\,09{,}8
\end{aligned}$$

§ 112. *Problem* 3. Given, a, C, b; to find c.

Formula Y, No. 7 and 8.

$$\tan Z = \frac{2 \sin \tfrac{1}{2} C \, (a\,b)^{\frac{1}{2}}}{a \backsim b} \quad ; \quad c = \frac{a \backsim b}{\cos Z}$$

$a = 4539,3$ log $= 3,6569889$

$b = 3745,37$ log $= 3,5734947$

$7,2304836$

$(a\,b)^{\frac{1}{2}} =$ log $= 3,6152418$

$a \backsim b = 793,93$ $A:C:$ log $= 7.1002178$ log $= 2,8997822$

$\tfrac{1}{2} C = 24^\circ.\,45'.\,50''$ log sin $= 9.6220438$

2 log $= 0,3010300$

log tan $Z = 0.6385334$ $A:C:$ log cos $= 0.6497139$

$c = 3544,02$ . . . . log $= 3,5494961$

Given as above.

Formula Y, No. 10 and 11.

$$\cos x = \frac{2 \cos \tfrac{1}{2} C \, (a\,b)^{\frac{1}{2}}}{a + b} \quad ; \quad c = (c + b) \sin x$$

$a = 1966,26$ log $= 3,2936444$

$b = 3746,25$ log $= 3,5735930$

$6,8672374$

$(a\,b)^{\frac{1}{2}}$ . . . . . . log $= 3,4336187$

$a + b = 5712,51$ $A:C:$ log $= 6.2431730$ log $= 3,7568270$

$\tfrac{1}{2} C = 29^\circ.\,24'.\,15''$ log cos $= 9.9401069$

2 log $= 0,3010300$

log cos $x = 9.9179286$ log sin $= 9.7489740$

$c = 3204,8$ . . . . log $= 3,5058010$

113. *Problem* 4. Given, a, b, c; to find B.

Formula Y, No. 15.

$$\sin \tfrac{1}{2} B = \left(\frac{(p-a)(p-c)}{a\,c}\right)^{\frac{1}{2}} \quad ; \quad p = \frac{a+b+c}{2}$$

b =	1920,6		
a =	3409,3	ar.co.log =	6.4673348
c =	2591,6	ar.co.log =	6.5864320
	7921,5		
p =	3960,75		
$p-a$ =	551,45	log =	2,7415061
$p-c$ =	1369,15	log =	3,1364510
			18.9317239
$\frac{1}{2} B$ =	16°. 59'. 49",5	log sin =	9.4658619
B =	33°. 59'. 39".		

Given as above.

Formula Y, No. 16.

$$\cos \tfrac{1}{2} B = \left(\frac{p\,(p-b)}{a\,c}\right)^{\frac{1}{2}}$$

b =	2587,4		
a =	2468,8	ar. co. log =	6.6075141
c =	1584,2	ar. co. log =	6.8001900
	6640,4		
p =	3320,2	log =	3,5211642
$p-b$ =	732,8	log =	2,8649855
			19.7938538
$\frac{1}{2} B$ =	37°. 56'. 00".	log cos =	9.8969269
B =	75°. 52'. 00".		

Z

Given as above.

Formula Y, No. 17.

$$\tan \tfrac{1}{2} B = \left(\frac{(p-a)\,(p-c)}{p\,(p-b)}\right)^{\frac{1}{2}}$$

$$\begin{array}{rcll}
b & = & 2325{,}2 & \\
c & = & 3106{,}4 & \\
a & = & \underline{2459{,}8} & \\
 & & 7891{,}4 & \\
p & = & 3945{,}7 & \text{ar . co . log} = 6.4038759 \\
p - b & = & 1620{,}6 & \text{ar . co . log} = 6.7903510 \\
p - c & = & 839{,}3 & \text{log} = 2{,}9239172 \\
p - a & = & 1485{,}9 & \text{log} = \underline{3{,}1719896} \\
 & & & \phantom{\text{log} = {}}19.2901337 \\
\tfrac{1}{2} B & = & 23^{\circ}.\ 49'.\ 41'',1 & \text{tan} = 9.6450668 \\
B & = & 47^{\circ}.\ 39'.\ 22'',2 &
\end{array}$$

§ 114. *Problem* 5. Given, C, c, b; to find a.

Formulæ Y, No. 19, 20, 21.

$$\sin B = \frac{b \sin C}{c} \quad ; \quad \tan y = \left(\frac{c \cos B}{b \cos C}\right)^{\frac{1}{2}} \quad ; \quad a = \frac{b \cos C}{\cos^2 y}$$

$c =$ 3106,4	ar . co . log	= 6.5077426	log	= 3,4922574			
$b =$ 2468,2	log	= 3,3923803	ar . co . log	= 6.6076197		log	= 3,3923803
$C =$ 50°. 21'. 14"	log sin	= 9.8864910	ar . co . log cos	= 0.1951495		log cos	= 9.8048505
	log sin B	= 9.7866139	log cos	= 9.8981809			
				= 20.1932075			
			$\tan y$	= 0.0966037	ar . co . log cos²		0,2041452 0,2041452
$a =$ 4032,01						log	= 3,6055212

§ 115. Calculations of the surface of the triangle *ABC*, whose sides are *a*, *b*, *c*.

***Problem* 1. Given, *B*, *C*, *a*.**

Formula Z, No. 1 ; $S = \dfrac{a^2 \sin B \sin C}{2 \sin (B + C)}$

$$\begin{array}{rcll}
B & = & 52^\circ\ 58'\ 50'' & \log \sin = 9.9022375 \\
C & = & 64.\ 11.\ 10 & \log \sin = 9.9543454 \\
B + C & = & 117.\ 10.\ 00 & \text{ar.co.log:sin} = 0.0507651 \\
a & = & 2468{,}9 & 2 \log = \left\{ \begin{array}{l} 3{,}3925145 \\ 3{,}3925145 \end{array} \right. \\
 & & 2 & \text{ar.co.log} = 9.6989700 \\
S & = & 2462334, & \log = 6{,}3913470
\end{array}$$

§ 116. *Problem* 2. Given, *a*, *C*, *b*.

Formula Z, No. 3 ; $S = \dfrac{a \,.\, b \,.\, \sin C}{2}$

$$\begin{array}{rcll}
a & = & 3007{,}2 & \log = 3{,}4781630 \\
b & = & 2092{,}85 & \log = 3{,}3207381 \\
C & = & 89^\circ.\ 54'.\ 50'' & \log \sin = 9.9999995 \\
 & & 2 & \text{ar.co.log} = 9.6989700 \\
S & = & 3146810, & \log = 6{,}4978706
\end{array}$$

§ 117. *Problem* 3. Given, *a*, *b*, *c*; to find *S*.

Formula Z, No. 5.

$$S = (p\,(p - a)\,(p - b)\,(p - c))^{\frac{1}{2}}$$

$$p = \frac{a + b + c}{2}$$

a = 3330,4
b = 2965,9
c = 2325,3

8621,6
p = 4310,8 log = 3,6345579
$p - a$ = 980,4 log = 2,9914033
$p - b$ = 1344,9 log = 3,1286900
$p - c$ = 1985,5 log = 3,2978699

13,0525211
S = 3359390 log = 6,5262605

CHAPTER III.

Calculations of Spherical Trigonometry.

§ 118. After what has been said of the methods of calculation in the preceding chapter, it is not considered necessary to enter into the detail of the actual calculation of the formulæ of Right Angled Spherical Trigonometry, that are contained in series b. It may be observed, that they all require no more than the addition of two logarithms of trigonometric functions, in a manner exactly analogous to section 108, with this difference alone, that all the factors are trigonometric functions. Hence it is also evident, that relations only are obtained, not absolute quantities, as is the fact; for as we have only functions resulting from the relations of lines, no absolute quantity, or lineal dimension, can be in the result. This is the great means by which the relations of the immense and immeasurable distances that astronomy calculates, are obtained. When it becomes necessary to indicate real determinate magnitudes, as, for instance, in relation to the earth, it is evident, that the radius, which otherwise forms no element of the calculation, comes into consideration. In that case, it is necessary to multiply any result or formula, prepared for this purpose, by the value of the radius, expressed in that kind of unity in which it is wished to obtain the expression; in the first power when a mere lineal dimension is desired; in the square when a surface is required; and in the cube when a solid. This is exactly analogous to what has been said (section 11) in respect to right angled plane triangles; and all the formulæ of series Y, and Z, are examples of the same principle, as observed in sections 58, and 65; it applies equally to all the formulæ that follow hereafter.

We may proceed to the calculation of the formulæ of Oblique Angled Spherical Trigonometry, which require, of course, more arrangement and attention. As they are all expressly formed so as to admit of calculation by logarithms throughout, we shall dispense with the notation *log* before the trigonometric functions named; and consider it as always understood, that the logarithm of the trigonometric function indicated is used.

§ 119. *Problem* 1. Given, *b*, *B*, *c*; to find *C*.

$$\text{Formula g} \quad ; \quad \sin C = \frac{\sin B \sin c}{\sin b}$$

$$\begin{array}{llllll}
b & = & 80^\circ 41' 45'' & \text{ar.co.sin} & = & 0.0057515 \\
c & = & 79.\ 40.\ 09. & \sin & = & 9.9929018 \\
B & = & 83.\ 39.\ 59. & \sin & = & 9.9973412 \\
\cline{6-6}
C & = & 82.\ 13.\ 49. & \sin & = & 9.9959945
\end{array}$$

§ 120. *Problem* 2. Given, *B*, *c*, *C*; to find *b*.

$$\text{Formula h} \quad ; \quad \sin b = \frac{\sin B \sin c}{\sin C}$$

$$\begin{array}{llllll}
C & = & 40^\circ 51' 16'' & \text{ar.co.sin} & = & 0.1843305 \\
B & = & 29.\ 14.\ 12. & \sin & = & 9.6887918 \\
c & = & 39.\ 10.\ 04. & \sin & = & 9.8004375 \\
\cline{6-6}
b & = & 28.\ 08.\ 14. & \sin & = & 9.6735598
\end{array}$$

§ **121.** ***Problem*** **3. Given, a, b, c; to find A.**

Formula i, No. 1

$$\text{in}\ \tfrac{1}{2}A = \left(\frac{\sin(p-c)\sin(p-b)}{\sin b \sin c}\right)^{\frac{1}{2}} \quad ; \quad p = \frac{a+b+c}{2}$$

$$\begin{array}{rcll}
a & = & 73^\circ\, 39'\, 59'' & \\
b & = & 84.\,09.\,58. & \text{ar.co.sin} = 0.0022551 \\
c & = & 60.\,15.\,13. & \text{ar.co.sin} = 0.0613652 \\
\hline
 & & 218.\,05.\,10. & \\
p & = & 109.\,02.\,35. & \\
p-c & = & 38.\,47.\,22. & \sin = 9.7968935 \\
p-b & = & 24.\,52.\,37. & \sin = 9.6239464 \\
 & & & \overline{19.3844602} \\
\tfrac{1}{2}A & = & 29.\,29.\,30{,}8 & \sin = 9.6922301 \\
A & = & 58.\,59.\,01{,}6 &
\end{array}$$

Given as above.

Formula i, No. 2.

$$\cos \tfrac{1}{2}A = \left(\frac{\sin p \sin(p-a)}{\sin b \sin c}\right)^{\frac{1}{2}}$$

$$\begin{array}{rcll}
a & = & 98^\circ\, 42'\, 03'' & \\
b & = & 83.\,32.\,26. & \text{ar.co.sin} = 0.0027658 \\
c & = & 45.\,48.\,03. & \text{ar.co.sin} = 0.1495004 \\
\hline
 & & 227.\,02.\,32. & \\
p & = & 113.\,31.\,16. & \sin = 9.9623282 \\
p-a & = & 14.\,49.\,13. & \sin = 9.4078800 \\
 & & & \overline{19.5224744} \\
\tfrac{1}{2}A & = & 54.\,45.\,16. & \cos = 9{\cdot}7612372 \\
A & = & 109.\,30.\,32. &
\end{array}$$

Given as above.

Formula i, No. 3.

$$\tan \tfrac{1}{2}A = \left(\frac{\sin(p-c)\sin(p-b)}{\sin p \sin(p-a)}\right)^{\frac{1}{2}}$$

$$
\begin{array}{rcllcl}
a & = & 89^\circ\,14'\,16'' & & & \\
b & = & 73.\,52.\,43. & & & \\
c & = & 67.\,24.\,15. & & & \\
\hline
 & & 230.\,31.\,14. & & & \\
p & = & 115.\,15.\,37. & \text{ar . co . sin} & = & 0.0436497 \\
p-a & = & 26.\,01.\,21. & \text{ar . co . sin} & = & 0.3579096 \\
p-b & = & 41.\,22.\,54. & \text{sin} & = & 9.8202487 \\
p-c & = & 47.\,51.\,22. & \text{sin} & = & 9.8700887 \\
\hline
 & & & & & 19.0918877 \\
\frac{1}{2}\mathcal{A} & = & 19.\,21.\,03{,}15 & \text{tan} & = & 9.5459488 \\
\mathcal{A} & = & 38.\,42.\,06{,}3 & & &
\end{array}
$$

§ 122. ***Problem.* 4. Formulæ k.**

These formulæ having evidently the same form as those of the preceding problem, the arrangement for calculation is precisely similar; it is therefore unnecessary here to give any examples. The only difference between them is, that they use the cosines instead of the sines; and that the factors alternate between the formulæ for the sine and the cosine; and consequently appear in inverse order in the formula for the tangent.

§ 123. ***Problem* 5. Given, b, c. $\mathcal{A}$; to find a.**

Formulæ 1, No. 2 and 3.

$$\tan Z = \frac{\sin\frac{1}{2}\mathcal{A}\,(\sin b \sin c)^{\frac{1}{2}}}{\sin\frac{1}{2}\,(c \backsim b)} \quad ; \quad \sin\tfrac{1}{2}\,a = \frac{\sin\frac{1}{2}\,(c \backsim b)}{\cos Z}$$

$$
\begin{array}{rcllcllcl}
b & = & 89^\circ 14'\,18'' & \text{sin} & = & 9.9999616 & & & \\
c & = & 17.\,07.\,15. & \text{sin} & = & 9.4689198 & & & \\
\hline
b-c & = & 72.\,07.\,03. & & & 19.4688814 & & & \\
\hline
(\sin b \sin c)^{\frac{1}{2}} & = & \ldots\ldots\ldots & & & 9.7344407 & & & \\
\frac{1}{2}(b-c) & = & 36.\,03.\,31{,}5 & \text{ar . co . sin} & = & 0.2301690 & & \text{sin} = & 9.7698309 \\
\frac{1}{2}\mathcal{A} & = & 42.\,16.\,08. & \text{sin} & = & 9.8277639 & & & \\
\hline
 & & & \tan Z & = & 9.7923736 & \text{ar . co . cos} & = & 0.0706257 \\
\hline
\frac{1}{2}\,a & = & 43.\,49.\,58{,}8 & \ldots\ldots\ldots & & & & \text{sin} = & 9.8404566 \\
a & = & 87.\,39.\,57{,}6 & & & & & &
\end{array}
$$

Formulæ 1, No. 5, 6, 8, 9, 11, and 12.

These being all of the same form as the preceding, using only other trigonometric functions of the same data, they come under the same form of calculation, the other functions taking the place of those used in the above formula, each for each, in its respective place. It is on this account not necessary to give examples of them.

Given as above.

Formulæ 1, No. 8, 11, 14. No. 13 is calculated upon the same form.

$$\sin Z'' = \frac{\cos \tfrac{1}{2} A \, (\sin b \sin c)^{\frac{1}{2}}}{\sin \tfrac{1}{2}(b + c)}$$

$$\tan Z''' = \frac{\cos \tfrac{1}{2} A \, (\sin b \sin c)^{\frac{1}{2}}}{\cos \tfrac{1}{2}(b + c)}$$

$$\tan \tfrac{1}{2} a = \tan \tfrac{1}{2}(b + c) \cos Z'' \cos Z'''$$

$$\begin{array}{rll}
b = 89^\circ\, 14'\, 18'' & \sin = 9.9999616 & \\
c = 17.07.15. & \sin = 9.4689198 & \\
\hline
b+c = 106.21\ 33. & \quad 19.4688814 & \\
 & (\sin b \sin c)^{\frac{1}{2}} = 9.7344407 = m & \\
\tfrac{1}{2} A = 42.16.08. & \cos = 9.8692220 = n & \\
\tfrac{1}{2}(b+c) = 53.10.46{,}5 & \left\{\begin{array}{l}\text{ar.co.sin} = 0.0966290 = p \\ \text{ar.co.cos} \quad 0.2223493 = q\end{array}\right\} & \tan = 0.1257203 \\
 & \sin Z'' = 9.7002917 = m+n+p; & \cos = 9.9370883 \\
 & \tan Z''' = 9.8260120 = m+n+q; & \cos = 9.9195002 \\
\tfrac{1}{2} a = 43.50.60. & \ldots\ldots\ldots\ldots & \tan \tfrac{1}{2} a = 9.9823088 \\
a = 87.40.00. & &
\end{array}$$

Given as above.

Formulæ 1, No. 15 and 16.

$$\tan y = \cos A \tan b$$

$$\cos a = \frac{\cos b \cos (c \backsim y)}{\cos y}$$

$$
\begin{array}{rcl l l}
A & = & 84^\circ 32' 16'' & \cos = 8.9785888 & \\
b & = & 89.\ 14.\ 18. & \tan = \underline{1.8763321} & \cos = 8.1236295 \\
y & = & 82.\ 02.\ 57{,}8 & \tan y = 0.8549209 & \text{ar.co.cos} = 0.8591168 \\
c & = & \underline{17.\ 07.\ 15.} & & \\
c \backsim y & = & 64.\ 55.\ 42{,}8 & \ldots\ldots\ldots & \cos = \underline{9.6271077} \\
a & = & 87.\ 39.\ 57{,}5 & \ldots\ldots\ldots & \cos = 8.6098540
\end{array}
$$

Given as above (but with one angle obtuse.)

Formulæ I, No. 15 and 16.

$$
\begin{array}{rcl l l}
A & = & 121^\circ 36' 19'',8 & \cos = 9.7193874 - & \\
b & = & 50.10.30. & \tan = \underline{0.0788818} & \cos = 9.8064817 \\
y & = & 147.51.10. & \tan = 9.7982692 - \quad A:C:\cos & = 0.0722788 - \\
c & = & \underline{40.00.10.} & & \\
c \backsim y & = & 107.51.00. & \ldots\ldots\ldots & \cos = \underline{9.4864674 -} \\
a & = & 76.35.36. & \ldots\ldots\ldots & \cos = 9.3652279 +
\end{array}
$$

The effect of the obtuse angle at A, will be observed here, its cosine becomes negative; this is indicated by placing the sign — at the end; in consequence of which also, tan y becomes negative; the obtuse angle is therefore to be taken for y, in consequence of which its cosine also becomes negative; and the angle $c \backsim y$ becoming again negative, the last calculation presents two — signs, which producing again +, give for a an acute angle. This mode of accounting for the effect of the signs entirely obviates all difficulties.

The formulæ No. 17 and 18 being of the same form as the above, these examples will also serve for them.

§ 124. *Problem* 6. The formulæ of this problem, or series m, are all of the same form as the foregoing; the examples for calculation are to be arranged in the same manner, each respectively as its corresponding one.

§ 125. *Problem* 7. Given, C, a, b; to find A, and B.

Formulæ n, No. 1 and 2.

$$\tan \tfrac{1}{2}(A + B) = \cot \tfrac{1}{2} C \, \frac{\cos \tfrac{1}{2}(a \backsim b)}{\cos \tfrac{1}{2}(a + b)}$$

$$\tan \tfrac{1}{2}(A \backsim B) = \cot \tfrac{1}{2} C \, \frac{\sin \tfrac{1}{2}(a \backsim b)}{\sin \tfrac{1}{2}(a + b)}$$

$$\begin{array}{rll}
a = & 62^\circ\,25'\,32'' & \\
b = & \underline{43.\,19.\,11.} & \\
a + b = & 105.\,44.\,43. & \\
a \backsim b = & 19.\,06.\,21. & \\
\tfrac{1}{2}(a + b) = & 52.\,52.\,21{,}5 \quad \text{ar.co.cos} = 0.2192519 & \text{ar.co:sin} = 0.0983805 \\
\tfrac{1}{2}(a \backsim b) = & 9.\,33.\,10{,}5 \quad \cos = 9.9939354 & \sin = 9.2199993 \\
\tfrac{1}{2} C = & 42.\,05.\,10. \quad \cot = \underline{0.0442502} & \cot = \underline{0.0442502} \\
\tfrac{1}{2}(A + B) = & 61.\,04.\,00{,}7 \quad \tan = 0.2574375 & \tan = 9.3626300 \\
\tfrac{1}{2}(A \backsim B) = & \underline{12.\,58.\,44{,}5} & \\
A = & 74.\,02.\,45{,}2 & \\
B = & 48.\,05.\,16{,}2 &
\end{array}$$

§ 126. *Problem* 8. The formulæ o, No. 1 and 2, being exactly of the same form as the foregoing, the same example may serve as a type for them.

§ 127. *Problem* 9.

Formulæ p, No. 1 and 2.

$$\tan y = \tan b \cos A \quad ; \quad \cos(c \backsim y) = \frac{\cos a \cos y}{\cos b}$$

$$\begin{array}{rll}
A = & 32^\circ\,10'\,15'' \quad \cos = 9.9276086 & \\
b = & 57.\,12.\,03. \quad \tan = \underline{0.1908206} & \text{ar.co:cos} = 0.2662444 \\
y = & 52.\,43.\,01{,}3 \quad \tan = 0.1184292 & \cos = 9.7822948 \\
 & \qquad a = 46^\circ\,17'\,12'' & \cos \equiv \underline{9.8395099} \\
c \backsim y = & \underline{39.\,23.\,48{,}9} \quad . \quad . \quad . \quad . & \cos = 9.8880491 \\
c = & 92.\,06.\,50{,}2 \quad \text{or} = 13.\,19.\,12{,}4 &
\end{array}$$

The same formulæ, with an obtuse angle at A.

$$\begin{array}{rlll}
A = & 121.36.19{,}8 & \cos = 9.7193874\ - & \\
b = & 50.10.30. & \tan = \underline{0.0788818} & \text{ar.co.}\cos = 0.1935183 \\
y = & 147.51.10. & \tan = 9.7902692\ - & \cos = 9.9277212\ - \\
 & & a = 76^\circ\ 35'\ 36' & \cos = \underline{9.3652279} \\
c \backsim y = & \underline{107.51.00.} & .\ \ .\ \ .\ \ .\ \ . & \cos = 9.4864674\ - \\
c = & 40.00.10. & &
\end{array}$$

Here the final result becomes a negative cosine, which therefore belongs to an obtuse angle, and produces c, acute, by the subtraction from the greater negative.

Given as above.

Formulæ p, No. 5, 6, 7, 8.

$$\sin B = \frac{\sin A \sin b}{\sin a} \quad ; \quad \tan x = \cos A \tan b$$

$$c = x \mathbin{\substack{+\\ \backsim}} y \quad ; \quad \tan y = \cos B \tan a$$

$$\begin{array}{lllll}
A = 32^\circ\ 10'\ 15'' & \sin = 9.7262756 & .\ .\ .\ .\ .\ . & & \cos = 9.9276066 \\
b = 57.12.03. & \sin = 9.9245762 & .\ .\ .\ .\ .\ . & & \tan = \underline{0.1908206} \\
a = 46.17.12 & \text{ar.co.}\sin = \underline{0.1409781} & \tan = 0.0195121 & & \tan x = 0.1184292 \\
 & \sin B = 9.7918299 & \cos = \underline{9.8949994} & & \\
x = 52.43.01{,}3 & & \tan y = 9.9145115 & & \\
y = \underline{39.23.48{,}8} & & & & \\
c = 92.06.50.1 & \text{or} \quad 13.19.12{,}5 & & &
\end{array}$$

Given as above.

Formulæ p, No. 5, 11 12, 13.

$$\sin Z = \sin A \sin b \quad ; \quad \cos x = \frac{\cos b}{\cos Z}$$

$$c = x \mathbin{\substack{+\\ \backsim}} y \quad ; \quad \cos y = \frac{\cos a}{\cos Z}$$

$$\begin{array}{llll}
\mathcal{A} = 32^\circ\,10'\,15'' & \sin = 9.7262756 & & \\
b = 57.12.03. & \sin = \underline{9.9245762} & & \cos = 9.7337556 = m \\
 & \sin Z = 9.6508518 & \text{ar.co.}\cos & = 0.0485393 = n \\
 & a = 46^\circ\,17'\,12'' & & \cos = \underline{9.8395098} = p \\
x = 52.43.01{,}2 & & & \\
y = \underline{39.23.49.} & & & \cos x = 9.7822949 = m + n \\
c = 92.06.50{,}2 & \text{or} = 12.19.12{,}2 & & \cos y = 9.8880491 = p + n
\end{array}$$

§ 128. *Problem* 10. Here we have to repeat what has been said in problems 6, and 8; the formulæ of this problem, or of series q, take in calculation exactly the same form as those of *problem* 9; the examples of which, therefore, also serve for this problem.

§ 129. *Problem* 11. Given, B, $\mathcal{A}$, a; to find c.

Formulæ r, No. 1 and 2, or 3 and 4.

$$\tan x = \cos B \tan a \quad ; \quad \sin (c + x) = \tan B \cot \mathcal{A} \sin x$$

$$\begin{array}{llll}
a = 56^\circ\,13'\,53'' & \tan = 0.1748021 & & \\
B = 60.42.08. & \cos = \underline{9.6896184} & \tan = 0.2509420 & \\
x = 36.11.54{,}3 & \tan = 9.8644205 & \sin = 9.7712412 & \\
 & \mathcal{A} = 50^\circ\,41'\,15'' & \cot = \underline{9.9132069} & \\
c + x = \underline{59.31.28{,}5} & \quad . \quad . \quad . & \sin = 9.9354301 & \\
c = 23.19.34{,}2 & \text{or} = 95.43.22{,}8 & &
\end{array}$$

Given as above.

Formulæ r, No. 5, 6, 7, 8.

$$\sin b = \frac{\sin a \sin B}{\sin \mathcal{A}} \quad ; \quad \tan x = \tan a \cos B$$

$$c = x \underset{\backsim}{+} y \quad ; \quad \tan y = \tan b \cos \mathcal{A}$$

$$\begin{array}{llll}
a = 56^\circ\,13'\,53'' & \sin = 9.9197521 & \quad . \quad . \quad . \quad . \quad . & \tan = 0.1748021 \\
B = 60.42.08. & \sin = 9.9405605 & \quad . \quad . \quad . \quad . \quad . & \cos = \underline{9.6896184} \\
\mathcal{A} = 50.41.15. & \text{ar.co.}\sin = \underline{0.1114263} & \cos = 9.8017807 & \tan x = 9.8644205 \\
 & \sin b = 9.9717389 & \tan = \underline{0.4284988} & \\
x = 36.11.54{,}3 & & & \\
y = \underline{59.31.28{,}6} & & \tan y = 0.2302795 & \\
c = 95.43.22{,}9 & \text{or} = 23^\circ\,19'\,34''{,}3 & &
\end{array}$$

Given as above.

Formulæ r, No. 11, 12, 13.

$$\sin Z = \sin a \sin B \quad ; \quad \cos x = \frac{\cos a}{\cos Z} \quad ; \quad c = x \underset{\text{S}}{+} y$$

$$\sin y = \frac{\cot A \sin a \sin B}{\cos Z}$$

B = 60° 48′ 08″	sin = 9.9405605	sin = 9.9405605	
a = 56. 13. 53.	sin = 9.9197521	sin = 9.9197521	cos = 9.7449501
	sin Z = 9.8603126	ar . co . cos = 0.1619110	ar . co . cos = 0.1619110
A = 50. 41. 15.	. . . .	cot = 9.9132070	cos x = 9.9068611
		sin y = 9.9354306	
x = 36. 11. 54,8			
y = 59. 31. 28,9			
c = 95. 43. 22,9	or = 23° 19′ 34″,3		

§ 130. *Problem* 12. The formulæ of this problem having exactly the same form as those of *problem* 11, it is considered unnecessary to give examples of the calculation of series s.

THE END.

CORRECTIONS.

Page.	*Line.*	
15	16	After "called," add, or.
16	12 to 19	Between the fractions place a full stop (.) as sign of multiplication, instead of the comma (,).
—	21	"H" read, No.
17	penult.	in beginning, "3" read, 4.
18	10	Above "1 and 2" write, A, No.
26	1	"produced upon" read, produced on.
33	2	from below, $\frac{\sin a}{\sin b}$, read, $\frac{\sin a}{\cos a}$
35	15	"No. 4 and 9" read, No. 1 and 9.
38	3	Above "No. 1" in the margin, place, I.
39	5	"7" read, 8.

Page 54, at the bottom, add the following.

The formulæ 6, 7, and 8, give also, when divided by sine, or cosine, the following expressions for the tangent of the half angle by the tangent, and cotangent of the whole angle.

From No. 6:

$$\tan \tfrac{1}{2} a = \frac{(1 + \tan^2 a)^{\frac{1}{2}} - 1}{\tan a} = (1 + \cot^2 a)^{\frac{1}{2}} - \cot a \qquad 9$$

From No. 7:

$$\tan \tfrac{1}{2} a = \frac{\tan a}{(1 + \tan^2 a)^{\frac{1}{2}} + 1} = \frac{1}{(1 + \cot^2 a)^{\frac{1}{2}} + \cot a} \qquad 10$$

From No. 8:

$$\tan \tfrac{1}{2} a = \frac{(1 + \cot^2 a)^{\frac{1}{2}} + 1 - \cot a}{(1 + \cot^2 a)^{\frac{1}{2}} + 1 + \cot a} = \frac{(1 + \tan^2 a)^{\frac{1}{2}} + \tan a - 1}{(1 + \tan^2 a)^{\frac{1}{2}} + \tan a + 1} \qquad 11$$

Page.	*Line.*	
58	9	“$4 \cos b \quad 3$” read, $4 \cos b - 3$.
70	9	“$\sin^3 a$” read, $\sin^2 a$.
72	16	“$n \ldots\ldots n_8$” read, $n \ldots\ldots n_{10}$
75	13	The divisor of the fourth term read thus, $2 . 3 . 4 . 5 . 6 . 7$
86	5	In the divisor, “$(a \backsim c)$” read, $(a \backsim c)^2$
93	3	“BC” read, B, C.
103	11	“Pcd” read, PCD.
107	26	“Lemma 1” read, Lemma 1 and 2.
——	27	“bc” read, be.
108	16	“$2 r^2 < \llcorner R$” read, $2 r^2 < 2 \llcorner R$.
111	12	“DGF by EGF” read, EGF by DGF.
——	last	“DGF” read, EGF.
118	8	In the divisor, “$\cos c_{\prime\prime} \backsim \cos c_{\prime\prime}$” read, $\cos c_{\prime\prime} \backsim \cos c_{\prime}$
119	1	” ” “$\tan \frac{1}{2} C_{\prime\prime} \backsim C)$” read, $\tan \frac{1}{2} (C_{\prime\prime} \backsim C)$
123		Above the numbers in the margin, place, f.
126	4	“No. 14 and 15” read, No. 15 and 16.
130	14	“h” read, i.
——	—	“i” read, k.
135	5	“8” read, 3.
141	13	In the divisor, “$\cos A$” read, $\cos^2 A$.
147	9	“4” read, 3.
149	19	Place the “1” two lines lower.
151	6	“21” read, 2.
156	7 & 8	“$(a c)^{\frac{1}{3}}$” read, $(a c)^{\frac{1}{2}}$
157	4	Place $\sin B = \dfrac{b \sin C}{c}$ in the lower line.
——	5	“20” read, 21.
——	—	“19” read, 20.
——	—	Place the auxiliary in the lower line.
163	last	$\dfrac{\cos A \cos y'}{\cos B}$ read, $-\dfrac{\cos A \cos y'}{\cos B}$
164	4	“$\cos A \tan B$” read, $\cos a \tan B$
——	9	“$\cos (c \backsim x)$” read, $\cos (c \backsim x')$
169	26	“6.7039537” read, 6.7089537.
172	27	“logarithm” read, logarithms.
175	9	“20. 00. 39,3” read, 22. 00. 39,3.

[illegible]